Problem 1
Which of these matrices is totally unimodular? Justify your answer.

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

Problem 2
Let \( M \in \mathbb{Z}^{n \times m} \) be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. \( M^T \)
2. \( (M \ I_n) \)
3. \( (M \ -M) \)
4. \( M \cdot (I_n - 2e^T_j e_j) \) for some \( j \).

\( I_n \) is the \( n \times n \) identity matrix and \( e_j \) is the vector having a 1 in the \( j \)-th component, and 0 in the other components.

Problem 3
Given the weighted graph on the right, find the following:

(a) A matching which does not cover all vertices and has weight 15.

(b) A \( w \)-vertex cover of weight 16 where at least 7 vertices have non-zero weights.

Note: Given a weighted graph \( G = (V, E) \) with weight \( c \). A \( w \)-vertex cover of \( G \) is a weight distribution \( w : V \rightarrow \mathbb{R} \) on the vertices such that \( w(u) + w(v) \geq c(uv) \) for all edges \( uv \).

Def: The node-edge incidence matrix of a graph \( G = (V, E) \) is the matrix \( A \in \{0, 1\}^{V \times |E|} \) with

\[
A(v, e) = \begin{cases} 
1, & \text{if } v \in e, \\
0, & \text{otherwise.}
\end{cases}
\]

Problem 4
Let \( G \) be a cycle and let \( A \) be its node-edge incidence matrix. Give the possible values of \( \det(A) \) depending if \( G \) is an odd or an even cycle.
Problem 5
A family of sets $\mathcal{C} \subset 2^{[n]}$ is a chain if for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let $A \in \{0,1\}^{(|\mathcal{C}_1|+|\mathcal{C}_2|) \times n}$ with $A_{S,i} = 1$ if $i \in S$ and 0 otherwise, for $i = 1, \ldots, n$ and $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that $A$ is totally unimodular. *Hint: use induction on the size of a square submatrix of $A$. 