Problem 1

Consider the two player matrix game defined by
\[
\begin{pmatrix}
3 & 3 & -8 \\
-1 & 2 & -1
\end{pmatrix}
\]

Write down a linear program that computes the value of the game
\[
\max_{x \in X} \min_{y \in Y} x^T Ay
\]
and find a strategy \(x^* \in X\) that guarantees this value as an expected payoff for the row-player.

Hint: Use our own python implementation of the simplex algorithm if you do not want to compute the strategy by hand.

Solution:

Observe that the column player is always going to prefer the first column to the second one since her incentive is to minimize the "loss". Thus, the optimal row strategy \(x^*\) can be equivalently found by solving the LP (5.9) in the lecture notes, with \(A\) being the reduced matrix
\[
\begin{pmatrix}
3 & -8 \\
-1 & -1
\end{pmatrix}
\]

When rewriting this LP in the standard form we obtain:

\[
\begin{align*}
\max_{x_0} & \quad x_0 \\
\text{s.t.} & \quad x_0 - 3x_1 + x_2 \leq 0, \quad (1) \\
& \quad x_0 + 8x_1 + x_2 \leq 0, \quad (2) \\
& \quad x_1 + x_2 \leq 1, \quad (3) \\
& \quad -x_1 - x_2 \leq -1, \quad (4) \\
& \quad -x_1 \leq 0, \quad (5) \\
& \quad -x_2 \leq 0. \quad (6)
\end{align*}
\]

where a feasible basis \(\{1, 3, 5\}\) corresponds to the solution \((x_0, x_1, x_2) = (-1, 0, 1)\). By running the simplex algorithm we obtain that this is an optimal primal solution to the above LP, with the corresponding dual optimum \(\lambda = (7/11, 4/11, 0, 1, 0, 0)\). Thus, an optimal row strategy is \(x^* = (0, 1)^T\), an optimal column strategy is \(y^* = (7/11, 0, 4/11)^T\) and the value of the game is \((x^*)^T Ay^* = -1\).

Problem 2

Given a mixed row strategy \(\hat{x}\) and the following LP
\[
\min\{(\hat{x}^T A)y : \sum_j y_j = 1, \ y \geq 0\},
\]
argue the following: solving this LP with the Simplex method produces a pure strategy.

**Solution:**

Observe that the constraint matrix of the above LP has the full column rank, denote it with $m$, and the problem is clearly feasible. This gives that the polyhedron $P$ corresponding to the feasible region has vertices. Furthermore, observe $P \subseteq [0,1]^m$ which implies that the LP is bounded and the Simplex terminates at a vertex.

Finally, no vertex/feasible basis can be defined by all the constraints $y_j \geq 0$, $j \in [m]$ being active. Otherwise, one has $\sum_j y_j = 0$. By using the above observations we obtain that each vertex $v$ of $P$ is induced by an index $k \in [m]$, i.e., $v$ is the unique solution to the system:

$$\sum_j y_j = 1,$$

$$y_j = 0, \ \forall j \in [m], \ j \neq k,$$

so each vertex of $P$ is a pure strategy.

**Problem 3**

Prove Loomis’ Theorem, i.e., for any two-person zero-sum game specified by a matrix $A \in \mathbb{R}^{m \times n}$ show the following:

$$\max_x \min_j x^T A e_j = \min_y \max_i e_i^T Ay,$$

where $x$ ranges over all vectors in $\mathbb{R}_m^+$ with $1^T x = 1$, and an analogous statement holds for $y$. This theorem states that there is a pure best response.

**Solution:**

By using Problem 2 we have that for any fixed $\hat{x} \in \mathbb{R}^n$ one has

$$\min_j \hat{x}^T A e_j = \min \{(\hat{x}^T A) y : \sum_j y_j = 1, \ y \geq 0\},$$

and an analogous statement holds for $\max_i e_i^T A \hat{y}$ with $\hat{y} \in \mathbb{R}^n$. The minimax theorem gives the desired result:

$$\max_x \min_j x^T A e_j = \max_x \min_j x^T Ay = \min_y \max_x x^T Ay = \min_y \max_i e_i^T Ay.$$

**Problem 4**

A matrix $P \in \mathbb{R}^{n \times n}$ is *stochastic*, if $p_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{j=1}^n p_{ij} = 1 \text{ for all } i.$$

Use duality to show that a stochastic matrix has a non-negative left eigenvector $p \in \mathbb{R}^m_{\geq 0}$ associated to the eigenvalue 1, i.e. that the following system has a non-zero solution

$$p^T P = p^T, \ p \geq 0.$$

**Solution:**

Consider the LP:

$$\min -1^T y$$

s.t. $y^T (P - I) = 0^T,$

$$y \geq 0,$$
and its dual:

\[
\begin{align*}
\text{max} & \quad 0^T x \\
\text{s.t.} & \quad (P - I)x \leq -1.
\end{align*}
\]

We first show that the dual is infeasible. Assume the contrary, let \( \bar{x} \) be a feasible solution to the dual and \( j = \arg \min_{i \in [n]} \bar{x}_i \). Denote with \( P_j \) the \( j \)-th column of \( P \), from the definition of stochastic matrices we have that \( P_j \bar{x} \) is a convex combination of components of \( \bar{x} \). This means that \( P_j \bar{x} \geq \min_{i \in [n]} \bar{x}_i = \bar{x}_j \) and equivalently \( P_j \bar{x} - \bar{x}_j > -1 \). A contradiction to feasibility of \( \bar{x} \).

Given that the dual is infeasible, by duality the primal has to be either infeasible or unbounded. It is unbounded since 0 is a feasible solution. Thus, there exists \( \bar{y} \) such that \( \bar{y}^T (P - I) = 0^T, \bar{y} \geq 0 \) and \( \bar{y} \neq 0 \).

**Problem 5**

Give an example of a pair of (primal and dual) linear programs, both of which have infinite sets of optimal solutions.

**Solution:**

One can take the primal

\[
\begin{align*}
\text{max} \{0^T x : & \quad x_1 + 2x_2 \leq 0, \quad -x_1 - 2x_2 \leq 0\}
\end{align*}
\]

and its dual

\[
\begin{align*}
\text{min} \{0^T y : & \quad y_1 - y_2 = 0, \quad 2y_1 - 2y_2 = 0, \quad y \geq 0\}
\end{align*}
\]