Problem 1
Consider the two player matrix game defined by
\[
\begin{pmatrix}
3 & 3 & -8 \\
-1 & 2 & -1
\end{pmatrix}
\]
Write down a linear program that computes the value of the game
\[
\max_{x \in X} \min_{y \in Y} x^T Ay
\]
and find a strategy \( x^* \in X \) that guarantees this value as an expected payoff for the row-player.

Hint: Use our own python implementation of the simplex algorithm if you do not want to compute the strategy by hand.

Problem 2
Given a mixed row strategy \( \hat{x} \) and the following LP
\[
\min \{(\hat{x}^T A)y : \sum_j y_j = 1, \ y \geq 0\},
\]
argue the following: solving this LP with the Simplex method produces a pure strategy.

Problem 3
Prove Loomis’ Theorem, i.e., for any two-person zero-sum game specified by a matrix \( A \in \mathbb{R}^{m \times n} \) show the following:
\[
\max_{x} \min_{j} x^T Ae_j = \min_{y} \max_{i} e^T_i Ay,
\]
where \( x \) ranges over all vectors in \( \mathbb{R}^m_+ \) with \( 1^T x = 1 \), and an analogous statement holds for \( y \). This theorem states that there is a pure best response.

Problem 4
A matrix \( P \in \mathbb{R}^{n \times n} \) is stochastic, if \( p_{ij} \geq 0 \) for all \( i, j \in \{1, \ldots, n\} \) and
\[
\sum_{j=1}^{n} p_{ij} = 1 \text{ for all } i.
\]
Use duality to show that a stochastic matrix has a non-negative left eigenvector \( p \in \mathbb{R}^n_{\geq 0} \) associated to the eigenvalue 1, i.e., that the following system has a non-zero solution
\[
p^T P = p^T, \ p \geq 0.
\]

Problem 5
Give an example of a pair of (primal and dual) linear programs, both of which have infinite sets of optimal solutions.