Problem 1
Determine the dual program for the following linear programs:

1. \[
\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 - 7x_3 \\
\text{s.t.} & \quad x_1 + 3x_2 + 2x_3 = 4 \\
& \quad x_1 + x_2 \leq 8 \\
& \quad x_1 - x_3 \geq -15 \\
& \quad x_1, x_2 \geq 0 
\end{align*}
\]

2. \[
\begin{align*}
\text{min} & \quad 3x_1 + 2x_2 - 3x_3 + 4x_4 \\
\text{s.t.} & \quad 2x_1 - 2x_2 + 3x_3 + 4x_4 \leq 3 \\
& \quad x_2 + 3x_3 + 4x_4 \geq -5 \\
& \quad 2x_1 - 3x_2 - 7x_3 - 4x_4 = 2 \\
& \quad x_1 \geq 0 \\
& \quad x_4 \leq 0 
\end{align*}
\]

Problem 2
In the setting of the matrix-game described in Section 5.1 of the lecture notes, show that for \( A \in \mathbb{R}^{m \times n} \), one has
\[
\max_i \min_j A(i, j) \leq \min_j \max_i A(i, j).
\]

Problem 3
Consider the following linear program \( \max \{ c^T x : Ax \leq b \} \) and its dual \( \min \{ b^T y : A^T y = c, y \geq 0 \} \). Suppose that both programs are bounded and feasible. Let \( x_0 \) and \( y_0 \) be feasible solutions of the primal, respectively the dual linear program. Show that the following are equivalent:

(i) \( x_0 \) and \( y_0 \) are optimal solutions of the primal, respectively the dual.

(ii) \( c^T x_0 = b^T y_0 \).

(iii) If a component of \( y_0 \) is positive, the corresponding inequality in \( Ax \leq b \) is satisfied by \( x_0 \) with equality.

Problem 4
For each of the following assertions, provide a proof or a counterexample.

(i) An index that has just left the basis \( B \) in the simplex algorithm cannot enter in the very next iteration.

(ii) An index that has just entered the basis \( B \) in the simplex algorithm cannot leave again in the very next iteration.
Problem 5
We define two different norms on vectors. The infinity-norm is defined by $\|y\|_\infty = \max_i |y_i|$ and the 1-norm is defined by $\|y\|_1 = \sum_i |y_i|$.

Let $A$ be an $m \times n$ matrix and let $b \in \mathbb{R}^m$ be a vector. Consider the problem of minimizing $\|Ax-b\|_\infty$ over all $x \in \mathbb{R}^n$.

Suppose that $v$ is the optimal value of the problem.

(a) Let $p \in \mathbb{R}^m$ be a vector satisfying $\|p\|_1 \leq 1$ and $p^T A = 0$. Show that $p^T b \leq v$.

(b) To obtain the best possible lower bound of the form considered in (a), we construct the following linear program
\[
\begin{align*}
\max & \quad p^T b \\
p^T A &= 0 \\
\sum_{i=1}^m |p_i| &\leq 1.
\end{align*}
\]

Using strong duality, show that the optimal solution of this problem is equal to $v$.

Problem 6
Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points
\[S = \{ p \in \mathbb{Z}^n : 0 \leq p_i \leq B, \forall i = 1, \ldots, n \},\]
whose points are all colored blue but one, which is red. We have an oracle that, given $i \in \{1, \ldots, n\}$ and $\alpha \in \{0, \ldots, B\}$, tells us whether there exists a red point $x^* \in S$ with $x^*_i \leq \alpha$. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.