Problem 1
Show the “if” direction of the Farkas’ lemma: given \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \), if there exists a \( \lambda \in \mathbb{R}^m \geq 0 \) such that \( \lambda^\top A = 0 \) and \( \lambda^\top b = -1 \), then the system \( Ax \leq b \) is unfeasible.

Solution:
Suppose that there exists \( x^* \in \mathbb{R}^n \) such that \( Ax^* \leq b \). Then, since \( \lambda \geq 0 \), we have:
\[
\lambda^\top Ax^* \leq \lambda^\top b \implies 0 \leq -1,
\]
a contradiction.

Problem 2
A polyhedron \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) contains a line, if there exists a nonzero \( v \in \mathbb{R}^n \) and an \( x^* \in \mathbb{R}^n \) such that for all \( \lambda \in \mathbb{R} \), the point \( x^* + \lambda v \in P \). Show that a nonempty polyhedron \( P \) contains a line if and only if \( A \) does not have full column-rank.

Solution:
Assume that \( P \) contains a line \( \{ x \in \mathbb{R}^n : x = x^* + \lambda \cdot v, \lambda \in \mathbb{R} \} \). We claim that \( v \in \ker(A) \), i.e. for all rows \( a_i \) of \( A \) we have \( a_i^\top v = 0 \). Assume for contradiction that there is a row \( a_i \) with \( a_i^\top v \neq 0 \). Then we can choose \( \lambda \in \mathbb{R} \) such that \( a_i^\top x^* + \lambda a_i^\top v > b_i \) (namely such that \( |\lambda| > \frac{b_i - a_i^\top x^*}{a_i^\top v} \)). Thus for \( x := x^* + \lambda v \) we have \( x \notin P \) because \( a_i^\top x > b_i \). This is a contradiction to the fact that \( P \) contains the line \( \{ x \in \mathbb{R}^n : x = x^* + \lambda \cdot v, \lambda \in \mathbb{R} \} \).

Thus the kernel of \( A \) is not empty, and \( A \) does not have full column rank.

Conversely, if \( A \) does not have full column rank, let \( x^* \) be some feasible point of the polyhedron, and let \( v \) be a nonzero vector from the kernel of \( A \). Then \( x^* + \lambda \cdot v \in P \) for all \( \lambda \in \mathbb{R} \). Hence \( P \) contains a line.

Problem 3
Given \( x^* = (0 \ 1 \ 1)^T \in \mathbb{R}^3 \) and the vector \( d = (1 \ 1 \ -1)^T \in \mathbb{R}^3 \) decide if the ray \( \{ x^* + \lambda d : \lambda \in \mathbb{R}_{\geq 0} \} \) intersects the following hyperplanes while moving in the direction of \( d \). Give the order in which the trajectory passes the planes.

\[
P_1 = \{ x \in \mathbb{R}^3 : (1 \ 2 \ 3)x = 0 \} \quad P_2 = \{ x \in \mathbb{R}^3 : (3 \ 2 \ 1)x = 4 \}
\]
\[
P_3 = \{ x \in \mathbb{R}^3 : (1 \ 1 \ 1)x = 2 \} \quad P_4 = \{ x \in \mathbb{R}^3 : (0 \ 1 \ 3)x = -1 \}
\]

Solution:
The trajectory of \( x^* \) is given by the line \( \{ x^* + \delta d : \delta \geq 0 \} \) where a point in the trajectory moves further away from \( x^* \) if \( \delta \) becomes larger. To find the order in which \( x^* \) passes the planes we search the corresponding \( \delta_i \) for which \( x^* + \delta_i d \) is in the plane \( P_i \) or decide that such a \( \delta \) does not exist.
\[ P_1: (1\ 2\ 3)((0\ 1\ 1)^T + \delta(1\ 1\ -1)^T) = 5 + 0 \cdot \delta = 5 \neq 0 \text{ for all } \delta, \text{ so } x^* \text{ does not pass } P_1 \text{ since it moves parallel to it.} \]

\[ P_2: (3\ 2\ 1)((0\ 1\ 1)^T + \delta(1\ 1\ -1)^T) = 3 + 4\delta = 4 \text{ for } \delta = \frac{1}{4}. \]

\[ P_3: (1\ 1\ 1)((0\ 1\ 1)^T + \delta \cdot (1\ 1\ -1)^T) = 2 + \delta = 2 \text{ for } \delta = 0, \text{ so } x^* \text{ is already on } P_3. \]

\[ P_4: (0\ 1\ 3)((0\ 1\ 1)^T + \delta \cdot (1\ 1\ -1)^T) = 4 - 2\delta = -1 \text{ for } \delta = \frac{5}{2}. \]

The order in which \( x^* \) passes the planes is \( P_3, P_2, P_4 \). The plane \( P_1 \) will never be passed.

**Problem 4**

Provide a proof or counterexample to the following statement:

Let \( \max \{c^T x : x \in \mathbb{R}^n, Ax \leq b\} \) be a linear program with \( A \in \mathbb{R}^{m \times n} \) of full column rank. If \( B \) is an optimal basis, then all the components of \( \lambda_B \) are strictly positive.

**Solution:**

False. Consider the linear program

\[
\begin{align*}
\max & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 1 \\
& \quad -x_1 \leq 0 \\
& \quad -x_2 \leq 0
\end{align*}
\]

The feasible region of this LP is the triangle in \( \mathbb{R}^2 \) with vertices \((0, 0)\), \((1, 0)\), and \((0, 1)\). An optimal basis is \( \{1, 2\} \) with corresponding feasible basic solution \((0, 1)\). Here,

\[
\lambda^T \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = [1 \quad 1]
\]

which implies \( \lambda^T = [1 \quad 0] \).

**Problem 5**

Consider the following LP:

\[
\begin{align*}
\max & \quad 2x + 4y + 3z \\
\text{s.t.} & \quad 2x - 3y - z \leq 3, & (1) \\
& \quad -x + 6y + 4z \leq 5, & (2) \\
& \quad -x + 3y + 2z \leq 2, & (3) \\
& \quad -x \leq 0, & (4) \\
& \quad -y \leq 0, & (5) \\
& \quad -z \leq 0. & (6)
\end{align*}
\]

a) Given the basis \( B = \{1, 2, 6\} \), compute \( x^* \) with \( A_B x^* = b_B \).

b) Decide whether \( x^* \) is feasible.

c) Compute \( \lambda \in \mathbb{R}^3 \) with \( \lambda^T A_B = c^T \).

d) Decide whether \( B \) is an optimal basis.

**Solution:**
a) Calculate $A_B^{-1}$ and write $x^* = A_B^{-1}b_B$

$$x^* = \frac{1}{9} \begin{bmatrix} 6 & 3 & 6 \\ 1 & 2 & 7 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 33 \\ 13 \\ 0 \end{bmatrix}$$

b) For the feasibility of $x^*$ it is sufficient to see if it fulfills all inequalities not in the basis, i.e. in the set $\{3, 4, 5\}$.

$$\frac{1}{9}(-33 + 3 \cdot 13 + 2 \cdot 0) = \frac{6}{9} \leq 2 \quad (3)$$

$$\frac{1}{9}(-33 + 0 + 0) \leq 0 \quad (4)$$

$$\frac{1}{9}(0 - 13 + 0) \leq 0 \quad (5)$$

c) We reuse $A_B^{-1}$ as calculated in a) to get the equation $\lambda^T = c^T A_B^{-1}$

$$\lambda^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 6 & 3 & 6 \\ 1 & 2 & 7 \\ 0 & 0 & -9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 16 & 14 & 13 \end{bmatrix}$$

d) $B$ is optimal. To see this, extend the $\lambda$ found in c) to a $\lambda' \in \mathbb{R}^6$ by adding zeros to at the lines not in $B$. Now $\lambda' \geq 0$ and it fulfills the equation $\lambda'A = c^T$. By Definition 4.3 of the lecture notes, $B$ is an optimal basis.