Problem 1
Show the “if” direction of the Farkas’ lemma: given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, if there exists a $\lambda \in \mathbb{R}_0^m$ such that $\lambda^T A = 0$ and $\lambda^T b = -1$, then the system $Ax \leq b$ is unfeasible.

Problem 2
A polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ contains a line, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron $P$ contains a line if and only if $A$ does not have full column-rank.

Problem 3
Given $x^* = (0 1 1)^T \in \mathbb{R}^3$ and the vector $d = (1 1 -1)^T \in \mathbb{R}^3$ decide if the ray $\{x^* + \lambda d : \lambda \in \mathbb{R}\}$ intersects the following hyperplanes while moving in the direction of $d$. Give the order in which the trajectory passes the planes.

$$
\begin{align*}
P_1 &= \{x \in \mathbb{R}^3 : (1 2 3)x = 0\} & P_2 &= \{x \in \mathbb{R}^3 : (3 2 1)x = 4\} \\
P_3 &= \{x \in \mathbb{R}^3 : (1 1 1)x = 2\} & P_4 &= \{x \in \mathbb{R}^3 : (0 1 3)x = -1\}
\end{align*}
$$

Problem 4
Provide a proof or counterexample to the following statement:
Let $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ be a linear program with $A \in \mathbb{R}^{m \times n}$ of full column rank. If $B$ is an optimal basis, then all the components of $\lambda_B$ are strictly positive.

Problem 5
Consider the following LP:

$$
\begin{align*}
\text{max} & \quad 2x + 4y + 3z \\
\text{s.t.} & \quad 2x - 3y - z \leq 3, \quad (1) \\
& \quad -x + 6y + 4z \leq 5, \quad (2) \\
& \quad -x + 3y + 2z \leq 2, \quad (3) \\
& \quad x \leq 0, \quad (4) \\
& \quad -y \leq 0, \quad (5) \\
& \quad -z \leq 0. \quad (6)
\end{align*}
$$

a) Given the basis $B = \{1, 2, 6\}$, compute $x^*$ with $A_B x^* = b_B$.

b) Decide whether $x^*$ is feasible.

c) Compute $\lambda \in \mathbb{R}^3$ with $\lambda^T A_B = c^T$.

d) Decide whether $B$ is an optimal basis.