Problem 1
Let $K \subseteq \mathbb{R}^n$ and $v \in K$ an extreme point of $K$. Show that $v$ cannot be written as a convex combination of other points in $K$.

Solution:
Since $v$ is an "extreme point" there exists an inequality $a^T x \leq \beta$ valid for $K$ such that $\{v\} = K \cap \{x \in \mathbb{R}^n : a^T x = \beta\}$. We prove the statement by contradiction. Without loss of generality assume that $v$ can be written as a convex combination of two points $u, w \in K$, i.e. $v = \lambda u + (1 - \lambda)w$. We obtain that
$$\beta = a^T v = a^T (\lambda u + (1 - \lambda)w) = \lambda a^T u + (1 - \lambda) a^T w < \lambda \beta + (1 - \lambda) \beta = \beta,$$
a contradiction. We used that $a^T u < \beta$ since $a^T x \leq \beta$ is valid for $K$ and $u \in K$ and $a^T w < \beta$ ($v$ is the only point in $K$ satisfying $a^T x \leq \beta$ with equality). Analogously $a^T w < \beta$.

Problem 2
Find a counterexample (and argue why it is one) for Theorem 3.10 when (1) $K$ is convex but not closed, (2) $K$ is not convex but closed

Solution:
For (1), consider the set $K = \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$ and $p = e_1 \notin K$. Since for any $\epsilon > 0$, $B(x, \epsilon)$ intersects $K$, it is clear that we cannot strictly separate $p$ from $K$.
For (2), consider the set $K = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ and take $p = 0$.

Problem 3
Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$, rank($A$) = $n$ and $b \in \mathbb{R}^m$. Let $x^* \in P$ and $A'x \leq b'$ be given as in the lecture, i.e., the sub-system of $Ax \leq b$ consisting of inequalities that are satisfied by $x^*$ with equality. Suppose that $x^*$ is not a vertex. We know already that this is equivalent to rank($A'$) < $n$. In this exercise, you will show that $P$ contains at least one vertex.

i) Show that there exists a $d \in \mathbb{R}^n$ with $d \neq 0$ and $A'd = 0$.

ii) With this $d$, show that the line $\{x^* + \lambda d : \lambda \in \mathbb{R}\}$ is not contained in $P$.

iii) Deduce that there exists a feasible point $y^*$ of $P$ whose sub-system $A''x \leq b''$ of inequalities that are satisfied by $y^*$ with equality, satisfies rank($A''$) > rank($A'$).

iv) Conclude that $P$ has a vertex.

Solution:
$A'$ does not have full row rank (rank($A'$) < $n$), so we can find a $d \neq 0$ such that $A'd = 0$. If we denote by $A''$ the rows in $A$ but not in $A'$, we must have that $A''d \neq 0$: If all rows $a_r$ of $A''$ were orthogonal to $d$ (meaning $a_r d = 0$), then rank($A$) < $n$, contradiction. We call such a row $a''$. Since $a''d \neq 0$ and $A'd = 0$, we have that the row $a''$ is linearly independent from the rows in $A'$ (**).
Choosing now $\lambda$ big enough, we have that $x^* \pm \lambda d$ is not contained in $P$ (where we choose the sign depending on the sign of $a''d$). Conversely, since $A''x^* < b''$ and $A'd = 0$, we can choose $\epsilon > 0$ sufficiently small, such that $A(x^* \pm \epsilon d) \leq b$. For all rows $a_i \in A''$, denote by $\delta_i = b_i - a_ix^* > 0$. We can choose $\epsilon = \min |\delta_i|/d_i$ and we have that $A(x^* \pm \epsilon d) \leq b$ with at least one more row with equality (when taking the appropriate sign). This gives a new subsystem $A'_{\text{new}}$ that is satisfied by equality, by (*), $\text{rank}(A'_{\text{new}}) > \text{rank}(A')$. To conclude, we redo this procedure until we have found a vertex $v$ such that $Av = b$ for some subsystem $\tilde{A}$ of $A$ of rank equal to $n$.

Problem 4
Show the following: if $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and the system

$$Ax = b, \; x \geq 0 \tag{1}$$

admits a solution, then there is $\hat{x}$ such that $\hat{x}$ has only $m$ non-zero entries and is also a solution to (1).

Solution:
There are two ways to show this exercise, either by directly using the preceding exercise or by reusing the same idea for this new case. We first show the latter:

We show that if there is a solution with strictly more than $m$ non zero entries, then we can find a solution with at least one non zero entry less: Let $x$ be a solution with $l > m$ non-zero entries. We may suppose these correspond to the first $l$ columns of our matrix. Let $B \in \mathbb{R}^{m \times l}$ be the matrix consisting of the first $l$ columns of $A$. Since $l \geq m + 1$, this matrix $B$ does not have full row rank and so we can find $y$ such that $By = 0$. By adding $n - l$ zeroes to the vector $y$ (so passing from a vector in $\mathbb{R}^{l \times 1}$ to a vector in $\mathbb{R}^{n \times 1}$), we have that

$$Ay = 0$$

By construction, $y$ is only non-zero in the first $l$ entries - exactly where $x$ is strictly positive. So $x^* \pm cy$ is still a feasible solution for (1) for some small $c > 0$. Like in the previous exercise, we can take $c$ large enough and with the appropriate sign such that we can eliminate (at least) one non-zero coordinate of $y$ without destroying positivity.

The approach using the preceding exercise: Consider the following (equivalent) system:

$$\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} [x] \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

The new matrix has dimensions $\{(n + 2m) \times n\}$ and its rank is $n$ (because of the identity matrix we have attached at the end). By the preceding exercise, there exists a vertex $v$ defined by a subsystem $A'x = b'$ where the rank of $A'$ equals to $n$. But the rank of the matrix $\begin{pmatrix} A \\ -A \end{pmatrix}$ is smaller or equal to $m$. This shows that for this vertex $v$, at least $n - m$ inequalities of $-Ix \leq 0$ are satisfied with equality. This implies that $v$ has at most $m$ non zero entries.

Problem 5
A conic combination of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is a vector of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for each $i$. The set of all conic combinations of the $v_1, \ldots, v_k$ is denoted by $\text{cone}(\{v_1, \ldots, v_k\})$.

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of $A$.

i) Show that $\text{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\}$.
ii) Show that \( \text{cone}(\{a_1, \ldots, a_k\}) \) for \( k \leq n \) is the set

\[
P_k = \{ x \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \ldots, k, a_i^{-1}x = 0, i = k + 1, \ldots, n \},
\]

where \( a_i^{-1} \) denotes the \( i \)-th row of \( A^{-1} \).

Solution:

i) We obtain the following (where \([n]\) denotes the set \( \{1, 2, \ldots, n\} \)):

\[
\text{cone}\{a_1, \ldots, a_n\} = \{ y = \sum_{i \in [n]} \lambda_i a_i : \lambda_i \in \mathbb{R}_{\geq 0} \forall i \in [n] \} = \{ y = A\lambda : \lambda \in \mathbb{R}_{\geq 0}^n \} = \\
\{ y \in \mathbb{R}^n : A^{-1}y = \lambda, \lambda \geq 0 \} = \{ y \in \mathbb{R}^n : A^{-1}y \geq 0 \}.
\]

ii) Analogously one has:

\[
\text{cone}\{a_1, \ldots, a_k\} = \{ y = A\lambda : \lambda \in \mathbb{R}_{\geq 0}^n, \lambda_i = 0 \text{ for } i > k \} = \\
\{ y \in \mathbb{R}^n : A^{-1}y = \lambda, \lambda \geq 0, \lambda_i = 0 \text{ for } i > k \} = \\
\{ y \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \ldots, k, a_i^{-1}x = 0, i = k + 1, \ldots, n \}.
\]