Problem 1
A convex combination of the points $v_1, \ldots, v_k \in \mathbb{R}^n$ is a point of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ where $\lambda_i \geq 0$ for each $i$ and $\lambda_1 + \cdots + \lambda_n = 1$.
Let $K \subseteq \mathbb{R}^n$ and $v \in K$ an extreme point of $K$. Show that $v$ cannot be written as a convex combination of other points in $K$.

Problem 2
Find a counterexample (and argue why it is one) for Theorem 3.10 when (1) $K$ is convex but not closed, (2) $K$ is not convex but closed.

Problem 3
Consider a polyhedron $P = \{x \in \mathbb{R}^n; Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$, rank($A$) = $n$ and $b \in \mathbb{R}^m$. Let $x^* \in P$ and $A'x \leq b'$ be given as in the lecture, i.e., the sub-system of $Ax \leq b$ consisting of inequalities that are satisfied by $x^*$ with equality. Suppose that $x^*$ is not a vertex. We know already that this is equivalent to rank($A'$) < $n$. In this exercise, you will show that $P$ contains at least one vertex.

i) Show that there exists a $d \in \mathbb{R}^n$ with $d \neq 0$ and $A'd = 0$.

ii) With this $d$, show that the line $\{x^* + \lambda d; \lambda \in \mathbb{R}\}$ is not contained in $P$.

iii) Deduce that there exists a feasible point $y^*$ of $P$ whose sub-system $A''x \leq b''$ of inequalities that are satisfied by $y^*$ with equality, satisfies rank($A''$) > rank($A'$).

iv) Conclude that $P$ has a vertex.

Problem 4
Show the following: If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and the system

$$Ax = b, x \geq 0$$

admits a solution, then there exists a solution $\hat{x}$ that has only $m$ non-zero entries.

Hint: Use the previous exercise.

Problem 5
A conic combination of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is a vector of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for each $i$. The set of all conic combinations of the $v_1, \ldots, v_k$ is denoted by cone($\{a_1, \ldots, a_n\}$).
Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of $A$.

i) Show that cone($\{a_1, \ldots, a_n\}$) is the polyhedron $P = \{y \in \mathbb{R}^n; A^{-1} y \geq 0\}$.

ii) Show that cone($\{a_1, \ldots, a_k\}$) for $k \leq n$ is the set

$$P_k = \{y \in \mathbb{R}^n; a_i^{-1} x \geq 0, i = 1, \ldots, k; a_i^{-1} x = 0, i = k + 1, \ldots, n\},$$

where $a_i^{-1}$ denotes the $i$-th row of $A^{-1}$.