d-detecting and d-separating matrices

1 Motivation

Suppose you have 8 identical looking coins, one of which is counterfeit and lighter than the others. Using a double-pan scale (which only allows you to compare two weights), you would like to determine as quickly as possible which is the counterfeit coin. How many weighings are needed?

It is clear how to do it in 3 weighings, however, it is possible in 2!

Hint: For $3^n - 1$ genuine coins and 1 counterfeit coin, we need n weighings. Consider splitting the coins in three groups and only comparing the weight of two of them. What can you then conclude about the third?

2 d-detecting and d-separating matrices

Consider now the following generalization: Suppose we are given n identical looking coins and we know that at most d of them are counterfeit (lighter than the others). How many weighings (this time using a normal scale) are needed to detect which coins are genuine? This is equivalent to constructing a (d-separating) matrix M with entries 0 or 1, with k rows and n columns such that for all u, v in $\{0, 1\}^n$ with L^1 norm smaller or equal to d:

$$Mu = Mv \text{ if and only if } u = v$$

For each row r of M, $r \cdot u$ represents the weighing of a subset of the coins. It turns out, that we can choose $k = O(d \log(n))$, so only relatively few weighings are required to distinguish all counterfeit coins from the genuine coins.

Similarly, one can ask for a d-detecting matrix M (a 0/1 matrix of dimensions k times n) such that for all u, v in $\{0, 1, ..., d-1\}^n$, $u = v$ if and only if $Mu = Mv$. Also here, each row "corresponds" to a weighing of a subset of the coordinates. The naive way would be to weigh all coordinates separately - this corresponds to $M = I_n$, the identity matrix - but there is a way requiring only $O(n/\log_d(n))$ rows.

3 Aim of the project

In a first part of the project, the goal is to study some probabilistic constructions involving d-detecting/separating matrices. In particular, using the probabilistic method, one is able to construct (with high probability) a d-detecting/separating matrix with at most $O(d \log(n))$

and $O(n/\log_d(n))$ rows respectively and also with the probabilistic method one can prove that this is optimal, [2]. The goal is to understand this construction and the arguments involved.

The second part is partially open and depends on the interest of the student:

One direction would be to understand the deterministic construction of d-separating/detecting matrices. For $d = 2$, these can be constructed using elementary methods, [1], for bigger d, a construction using Fourier analysis is known, [3]. These constructions could also be implemented.

Another direction would be to understand the applications of d-detecting/separating matrices in the reconstruction of graphs or in integer programming.

4 Prerequisites

Very good knowledge of discrete mathematics, an interest in discrete optimization and a certain openness where this project leads us.

References

[1] Staffan Söderberg H. S. Shapiro, *A Combinatory Detection Problem*, The American Mathematical Monthly Vol. 70, No. 10 (Dec., 1963)

[2] V. Grebinski G. Kucherov, *Optimal Reconstruction of Graphs under the Additive Model*, Algorithmica (2000)

[3] N. Bshouty, *Optimal Algorithms for the Coin Weighing Problem with a Spring Scale*, available at https://pdfs.semanticscholar.org/e871/53626c8b32b39a5d419c1775b1871887390e.pdf