3rd Assignment

1. Show that, given an instance of the Steiner Tree problem, you can assume w.l.o.g. that the input graph is a metric graph.

2. Show that, given an instance of the Steiner Tree problem on a metric graph, the number of Steiner nodes in an optimal solution is \( \leq |R| - 2 \), where \( R \) is the set of terminals to be connected.

3. Give an exact algorithm for the Steiner Tree problem that runs in polynomial time if \(|R|\) is a fixed parameter (e.g. \(|R| = 5\)).

4. The aim of this exercise is to describe a bit-scaling technique that can be employed to derive an implementation of the 3/2-approximation algorithm for Steiner Tree problem on quasi-bipartite graphs \( G(R \cup X, E) \), with running time polynomial in the size of the input [R. Rizzi 1999].

Consider the sequence of costs \( c = c_0, c_1, \ldots, c_k \) where, for \( i > 0 \), \( c_i(e) := \lfloor \frac{c_i - 1(e)}{2} \rfloor \).

Let \( k \) be the smallest index for which \( c_k(e) \leq 1 \) for every edge \( e \). Note that \( k \leq \log_2 \{ \max \{ c(e) : e \in E \} \} \).

Still, for \( i = 0, 1, \ldots, k \) let \( T_i^* \) be an optimal Steiner tree and \( T_i \) be a \( \frac{3}{2} \)-optimal Steiner tree in \( G(R \cup X, E) \) with cost vector \( c_i \).

When the algorithm is executed on \( G(R \cup X, E) \) with costs \( c_k \) as input, then the “while loop” will cycle at most \( n \) times, since \( c_k \) is a 0, 1-vector. The output will be a \( \frac{3}{2} \)-optimal Steiner tree \( T_k \). Show that:

- \( c_{i-1}(T_i) \leq \frac{3}{2}n + \frac{3}{2}c_{i-1}(T_{i-1}^*) \).
- Using the previous result, show how to output a \( \frac{3}{2} \)-optimal Steiner tree for the original instance in polynomial time.