Optimization Methods in Finance	**(EPFL, Fall 2010)**

Lecture 8: Fundamental Theorem of Asset Pricing

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Financial background

In the last lecture we have been through basics of arbitrage which is summarized as follows.
An *arbitrage:* is a trading strategy that satisfies one of the following conditions:

- *Type A:* positive initial cash flow and no risk of loss later (Think of trading in terms of currency).
- *Type B:* no initial cash flow, no risk of loss and positive probability of making profit in future (for example a free lottery ticket).

To get in more details lets take an example of *European call options*. Recall that such an option is the right to buy at a given time, one stock of the underlying asset for the given strike price K. The question we want to answer here is how can we possibly price the derivative security such that no arbitrage possibilities are created. Let S_0 be the current price of the underlying stock (price at time 0). We will refer to the expiration date (i.e. the maturity) of the option as time 1. We make the simplifying assumption that at time 1 the price of stock is either in the *up state*

$$S_1^u = S_0 \cdot u \tag{1}$$

or in the *down state*

$$S_1^d = S_0 \cdot d \tag{2}$$

with $u > d$. Furthermore, let R be the risk-less rate of return for cash. Summarized, portfolios $x = (x_O, x_U, x_R)$ can be constructed as arbitrary linear combinations of

security \ cash flow	time 0	time 1 (up)	time 1 (down)
European Call option	$C_0 =$ **?**	$\max\{S_0 \cdot u - K, 0\}$	$\max\{S_0 \cdot d - K, 0\}$
Underlying asset	S_0	$S_0 \cdot u$	$S_0 \cdot d$
Cash	1	R	R

(here $x_0, x_U, x_C \in \mathbb{R}$ denote the amounts of options, stocks of the underlying and cash, respectively).

Remark 1 (*Negativity of portolio coefficient*). On first sight, it might look counterintuitive that we allow those quantities x_R, x_U, x_C to be negative. But in fact one can sell an option without owning the corresponding asset. In case that $x_O < 0$ and at time 1 the buyer of the option exercises it, we need to buy the asset from the market – no matter at what cost. Similarly $x_U < 0$ is meaningful since we can e.g. borrow an asset from a bank and sell it at time 0. At time 1 we then have to buy the underlying asset from the market to give it back to the bank.

To determine an appropriate price C_0 for the option, we make the following observation: For any two portfolios x and x' that have the same cash flow at time 1 (no matter whether we end up in the up state or the down state), the value of the portfolios at time 0 must be identical. This is the case, since otherwise selling the more pricy portfolio and buying the cheaper one provides type A arbitrage. This is the basic idea in the following approach.

Replication Strategy: (As stated in [1]) Can we form a portfolio x of the underlying security (long or short) and cash (borrowed or lent) today, such that the pay-off of the portfolio at the expiration date of the option will match the pay-off of the option? In other words, can we replicate the option using a portfolio of the underlying security and cash? Consider a portfolio x of Δ shares of underlying and B cash and R be the risk-less rate of return. At time 1 we have:
Up state:

$$\Delta \cdot S_0 \cdot u + B \cdot R = C_1^u \tag{3}$$

Down State:

$$\Delta \cdot S_0 \cdot d + B \cdot R = C_1^d \tag{4}$$

where $C_1^u = \max\{S_0 \cdot u - K, 0\}$ and $C_1^d = \max\{S_0 \cdot d - K, 0\}$. The situation is depicted below:

portfolio\cash flow	time 0	time 1 (up)	time 1 (down)
$(1,0,0)$	$C_0 = \mathbf{?}$	C_1^u	C_1^d
$(0,\Delta,B)$	$\Delta \cdot S_0 + B \cdot 1$	$\Delta S_0 u + B = C_1^u$	$\Delta S_0 d + BR = C_1^d$

Equation (3) and (4) yield the following results:

$$\Delta = \frac{C_1^u - C_1^d}{S_0 \cdot (u-d)} \tag{5}$$

$$B = \frac{u \cdot C_1^d - d \cdot C_1^u}{R \cdot (u-d)} \tag{6}$$

Substituting above results, the price of the portfolio today (that is at time 0) will be:

$$C_0 = \Delta S_0 + B = \frac{1}{R}\left(\left[\frac{R-d}{u-d}\right] C_1^u + \left[\frac{u-R}{u-d}\right] C_1^d\right) \tag{7}$$

The values $p_u = \frac{R-d}{u-d}$ and $p_d = \frac{u-R}{u-d}$ are called *risk neutral probabilities*. Observe that $p_u + p_d = 1$. We claim that $u > R > d$ and thus both probabilities are strictly positive. To see this claim, we consider the portfolios $(0,-1,S_0)$ and $(0,1,-S_0)$ which have the following cash-flows

portfolio\cash flow	time 0	time 1 (up)	time 1 (down)
$(0,-1,S_0)$	0	$S_0 \cdot (R-u)$	$S_0 \cdot (R-d)$
$(0,1,-S_0)$	0	$S_0 \cdot (u-R)$	$S_0 \cdot (d-R)$

If $R \geq u > d$, then the first portfolio provides type B arbitrage. If $u > d \geq R$, then the second portfolio provides type B arbitrage. Hence we can assume that $u > R > d$ (otherwise market allows arbitrage).

Fundamental theorem of asset pricing

Now let's review the theorem of complementary slackness which shall help us to prove the fundamental theorem of asset pricing.

Theorem 2 (Complementary Slackness)**.** *Consider the systems*

$$\begin{aligned} \max\{c^T x \mid Ax \le b\} \qquad & (P) \\ \min\{b^T y \mid A^T y = c, y \ge \mathbf{0}\} \qquad & (D) \end{aligned}$$

Then the following holds:

i) Let x^* be feasible for (P) and y^* be feasible for (D). Then x^* and y^* are both optimal solutions if and only if

$$y_i^* > 0 \Rightarrow (Ax^* - b)_i = 0$$

ii) There exist optimum solutions x^* for (P) and y^* for (D) such that

$$y_i^* > 0 \Leftrightarrow (Ax^* - b)_i = 0$$

Note that there are examples for LPs (P) and (D) where ii) does not hold for all optimum solutions.

Let $W = \{\omega_1, \dots, \omega_m\}$ be a finite set of possible future events for time 1. Now consider n assets with price S_0^i for $i = 1, \dots, n$ at time 0 and price $S_1^i(\omega_j)$ at time 1, if event ω_j occurred. Also note that we use $i = 0$ for the risk-less security.

Definition 3 ([1])**.** A *Risk neutral probability measure* on set $W = \{\omega_1, \dots, \omega_m\}$ is a set of positive numbers $p = (p_1, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$ such that for every security $i = 1, \dots, n$ we have:

$$S_0^i \cdot R = \sum_{j=1}^m p_j \cdot S_1^i(w_j) \triangleq \hat{E}\left[S_1^i\right] \tag{8}$$

Here $\hat{E}[S_1^i]$ is the expectation of S_1^i if the probability of event ω_j would be p_j.

Theorem 4. *A Risk neutral probability measure (RNPM) exists if and only if there is no arbitrage.*

Proof. Consider the following LP:

$$\min \sum_{i=1}^n S_0^i \cdot x_i \qquad (P) \tag{9}$$

$$\sum_{i=1}^n S_1^i(w_j) \cdot x_i \;\; \ge \;\; 0 \quad \forall j = 1, \dots, n \tag{10}$$

where x_i is the share of asset i. Suppose there is no arbitrage. This implies that the optimal solution of the LP is 0 (otherwise we had type A arbitrage). Now consider the dual of (P):

$$\begin{aligned} \min \sum_{j=1}^{m} 0 \cdot p_j \qquad & (D) \\ \sum_{j=1}^{m} p_j \cdot S_1^i\left(w_j\right) &= S_0^i \quad \forall i = 1, \ldots, n \\ p_j &\geq 0 \quad \forall j = 1, \ldots, m \end{aligned}$$

Since (P) is feasible and bounded, also (D) is feasible and bounded. Let x^* and p^* be a pair of optimum solutions for (P) and (D) satisfying ii) in Theorem 2. All inequalities in (P) must be satisfied by x^* with equality, otherwise x^* would provide type B arbitrage. As a consequence of Theorem 2 ii), one has $p_j^* > 0$ for all $j = 1, \ldots, m$. We have

$$\sum_{j=1}^{m} p_j^* \cdot \underbrace{S_1^0\left(w_j\right)}_{R} = \underbrace{S_0^0}_{1} \tag{11}$$

$$\Rightarrow \sum_{j=1}^{m} p_j^* \cdot R = 1 \tag{12}$$

hence $(p_i^* \cdot R, \ldots, p_m^* \cdot R)$ is a RNPM.

For the reverse direction, we assume to have a RNPM p^* and then show that there is no arbitrage. First note that is $\frac{p^*}{R}$ is a feasible solution for (D). Since all feasible solutions have the same objective value 0, this solution is also optimal. By duality, the primal LP has an optimum value of 0 as well. Hence there is no type A arbitrage. Next, let x^* be any optimum solution to (P). Combining Theorem 2 i) with the fact that $p_j^* > 0$, we obtain that $\sum_{i=1}^{n} S_1^i(w_j) \cdot x_i^* = 0$ holds for every $j = 1, \ldots, m$. This means there is also no type B arbitrage. The theorem is proven. $\square$

More details can be found in Section 4.1.3 (pages 74-75) in [1].

Application of LP in Arbitrage detection

The linear programming problem formulated above can be used to detect arbitrage opportunities. Consider a portfolio $x = (x_1, \ldots, x_n)$ of n derivatives depending on the same underlying with same maturity date. Let $S^i, i = 1, \ldots, n$ be their price at time 0 and $\Psi_i\left(S_1\right)$ be their prices at time 1 depending on price S_1 of the underlying at time 1. By K_i, we denote the strike price of option i. We assume that $K_1 < \ldots < K_n$. The function $\Psi_i\left(S_1\right)$ is piecewise linear in S_1 with one break point at K_i only. The price of the above portfolio at time 0 is $\sum_{i=1}^{n} x_i \cdot S^i$. The pay-off is given by

$$\Psi^x\left(S_1\right) = \sum_{i=1}^{n} x_i \cdot \Psi_i\left(S_1\right) \tag{13}$$

Clearly the pay-off function has n break points $K_1, \ldots, K_n$ as it is weighted sum of linear functions having one break point each. Now the pay-off function (piecewise linear function) is non-negative for $S_1 \geq 0$ if and only if following three conditions are met:

1. $\Psi^x(0) \geq 0$
2. $\Psi^x(K_i) \geq 0$
3. $\Psi^x(K_n+1) - \Psi^x(K_n) \geq 0$

Consider following linear program:

$$\begin{aligned} \min \sum_{i=1}^{n} x_i \cdot S_0^i & \\ \sum_{i=1}^{n} \Psi_i(0) \cdot x_i &\geq 0 \\ \sum_{i=1}^{n} \Psi_i(K_j) \cdot x_i &\geq 0 \quad \forall j = 1, \ldots, n \\ \sum_{i=1}^{n} \left[\Psi_i(K_n+1) - \Psi_i(K_n)\right] \cdot x_i &\geq 0 \end{aligned} \tag{14}$$

Theorem 5. *No type A Arbitrage $\Leftrightarrow$ optimal is zero. If no type A Arbitrage then no type B Arbitrage $\Leftrightarrow$ Dual has strictly positive solution.*

Hence we have following propositions (As stated in [1]):

- Proposition 1: There is no type-A arbitrage in prices S^i if and only if the optimal objective value of (14) is zero.
- Proposition 2: Suppose that there are no type A arbitrage opportunities in prices S^i. Then, there are no type B arbitrage opportunities if and only if the dual of the problem (14) has a strictly feasible solution.

The above propositions can be used to detect the arbitrage opportunities. Refer to section 4.2 on page 75 in [1] for a more detailed analysis.

References

[1] Gerard CORNUEJOLS & Reha TÜTÜNCÜ. Optimization Methods in Finance, *Cambridge University Press*, Cambridge (USA), 2007.