Reduction of convex optimization problem

Recall that the Convex Optimization Problem (C.O.P) is the following:

\[
\min \ f_0(x) \\
\text{s.t.} \ f_i(x) \leq 0 \quad \forall i = 1, \ldots, m \\
x \in \mathbb{R}^n
\]

where, \( f_0, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R} \) are convex functions.

This can be reduced to the decision problem, i.e, decide if the following set is feasible (not empty).

\[
Q_\delta := \{x \in \mathbb{R}^n | f_i(x) \leq 0, \forall i = 1, \ldots, m \text{ and } f_0(x) \leq \delta\}
\]

Binary search

Suppose \( L_0 \leq p^* \leq U_0 \) we can find \( p^* \) (approximately) by binary search, i.e,

1. Initialise \( L = L_0, U = U_0 \)
2. Repeat
   \[
   \delta := (U - L)/2
   \]
   If \( Q_\delta \neq \emptyset \) then \( U := \delta \)
   otherwise \( L := \delta \)

After \( k \) iterations, \( L \leq p^* \leq U \) and \( U - L = (U_0 - L_0)/2^k \).

Now we need to show that we can solve the decision problem. Assume, we are given the following:

- Bounded closed convex set \( K \subseteq \mathbb{R}^n \)
- \( L > 0 : \text{vol}(K) \geq L \) (\( K \) is full-dimensional)
- \( R > 0 : K \subseteq \{x \in \mathbb{R}^n | \|x\| \leq R\} \)
We must be able to solve separation problem. The separation problem is the following:

For any \( y \in \mathbb{R}^n \) we must be able to decide whether \( y \in K \) or \( y \notin K \) and if \( y \notin K \) return an hyperplane \( c^T x = \beta \) such that \( c^T x \leq \beta \), \( \forall x \in K \) and \( c^T y > \beta \).

**Definition 1.** The unit ball in \( \mathbb{R}^n \) is \( B = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \). Let \( f : \mathbb{R}^n \to \mathbb{R}^n : f(x) = Ax + b, A \in \mathbb{R}^{n \times n} \) regular, \( b \in \mathbb{R}^n \). Then the set

\[
E(A, b) = f(B) = \{ Ax + b \mid x \in B \}
\]

is termed an ellipsoid.

Note that \((A^{-1})^T A^{-1}\) is a positive definite, symmetric matrix.

The volume of an ellipsoid is:

\[
\text{vol}(E(A, b)) = |\text{det}(A)| \cdot \text{vol}(B) = |\text{det}(A)| \cdot \frac{1}{\pi^n} \left( \frac{2e\pi}{n} \right)^\frac{n}{2}
\]

where \( \frac{1}{\pi^n} \left( \frac{2e\pi}{n} \right)^\frac{n}{2} \) is an approximation of the volume of the unit ball.

**The ellipsoid method**

The goal for the *Ellipsoid method* is to find a point \( b \in K \). This is done as follows: One iteratively computes ellipsoids that always contain the set \( K \) fully, such that the volume of the ellipsoids decreases from iteration to iteration.

The initial ellipsoid \( E(A, b) \), is simply the ball of radius \( R \), i.e, \( A := RI, b = 0 \).

1. \( E(A, b) = \{ x \in \mathbb{R}^n \mid \|x\| \leq R \} \)
2. While \( \text{vol}(E(A, b)) \geq L \) Do
   3. If \( b \in K \) then RETURN \( b \)
   4. Compute separating hyperplane \( c^T x \leq \beta \)
      (i.e. \( c^T x \leq \beta \) \( \forall x \in K \) and \( c^T b > \beta \))
   5. Compute \( E(A', b') \supseteq E(A, b) \cap \{ x \mid c^T x \leq \beta \} \)
   6. Update \( E(A, b) := E(A', b') \)

In order to show that the ellipsoid method works, we must assure that \( \text{vol}(E(A', b')) \) is smaller than \( \text{vol}(E(A, b)) \).
Figure 1: Unit ball and ellipsoid $E(A', b')$

**Theorem 2.** For all $E(A, b) \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n / \{0\}$ there is an ellipsoid

$$E(A', b') \supseteq E(A, b) \cap \{x | c^T x \leq c^T b\}$$

such that

$$\frac{\text{vol}(E(A', b'))}{\text{vol}(E(A, b))} \leq e^{-\frac{1}{2(n+1)}}$$

Without loss of generality we can suppose:

- $b = 0$ by applying a translation to the ellipsoid
- $E(A, b) = B$ by a linear transformation
- $c = (-1, 0, \ldots, 0)$ by rotation

Define

$$E := E(A', b') : = E \left( \begin{pmatrix} \frac{n}{n+1} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{n^2}{n-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{\frac{n^2}{n-1}} \end{pmatrix}, \begin{pmatrix} \frac{1}{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

$$= \left\{ x \in \mathbb{R} \mid \|A'^{-1}(x - b)\|^2 \leq 1 \right\}$$

$$= \left\{ x \mid \left(\frac{n+1}{n}\right)^2 (x_1 - \frac{1}{n+1})^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}$$

**Lemma 3.** One has $B \cap \{x | x_1 \geq 0\} \subseteq E$ and

$$\frac{\text{vol}(E)}{\text{vol}(B)} \leq e^{-\frac{1}{2(n+1)}}$$

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Proof. Let \( x \in \mathbb{R}^n \) with \( \|x\| \leq 1 \) and \( x_1 \geq 0 \). Now we have:

\[
\|A'^{-1}(x-b)\|^2 = \left( \frac{n+1}{n} \right) \left( x_1 - \frac{1}{n+1} \right)^2 + \frac{n^2-1}{n} \sum_{i=2}^{n} x_i^2 \\
\leq \left( \left( \frac{n+1}{n} \right)^2 - \frac{n^2-1}{n} \right) x_1^2 - \left( \frac{2}{n+1} \left( \frac{n+1}{n} \right)^2 \right) x_1 + \left( \frac{n+1}{n} \right)^2 \frac{1}{(n+1)^2} + \frac{n^2-1}{n^2} \\
= \frac{2n+2}{n^2} x_1^2 - \frac{2n+2}{n^2} x_1 + 1 =: f(x_1)
\]

Since \( f \) is convex, the maximum of \( f(x_1) \) for \( 0 \leq x_1 \leq 1 \) must be attained for \( x_1 \in \{0, 1\} \). But \( f(0) = f(1) \), hence indeed \( B \cap \{x|x_1 \geq 0\} \subseteq E \).

The ratio of the volumes is:

\[
\frac{\text{vol}(E)}{\text{vol}(B)} = \|\det(A')\| = \frac{n}{n+1} \left( \frac{n^2}{n^2-1} \right)^{\frac{n-1}{2}}
\]

\[
\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}}
\]

\[
= e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}
\]

For inequality (3) we use \( 1 + x \leq e^x \forall x \in \mathbb{R} \) and for inequality (4) we use \( n^2 - 1 = (n+1)(n-1) \). □

Lemma 4. Let \( 0 \leq L \leq 1 \), the ellipsoid method finds a feasible point after at most \( k = 3n^2 \ln \left( \frac{2k}{L} \right) \) iterations.

Proof. After \( k := 2(n+1)n \ln \left( \frac{2R}{L} \right) \) iterations, we have:

\[
\text{vol}(E(A,b)) \leq R^n \text{vol}(B) \left( e^{-\frac{1}{2(n+1)}} \right)^k
\]

\[
\leq (2R)^n \left( e^{-\frac{1}{2(n+1)}} \right)^k
\]

\[
\leq (2R)^n e^{-n \ln \left( \frac{2R}{L} \right)}
\]

\[
= (2R)^n \frac{L^n}{|2R|^n} \leq L
\]

The last inequality comes from \( L \leq 1 \). □
Application to Mean Variance Optimization

Let us now outline, how the Ellipsoid method can be used to solve the Mean Variance Optimization problem. The goal is to find a point in

$$K := \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \forall i = 1, \ldots, n, \sum_{i=1}^{n} x_i \tilde{r}_i \geq r, x^T Q x \leq \delta \right\}$$

- As starting bounds for the binary search, we can choose $L_0 := 0, U_0 := \max_{i,j} Q_{ij}$.
- Including ball: $R = 1$ (because $\sum_{i=1}^{n} x_i = 1$)
- The question is how find a hyperplane to separate $y \in \mathbb{R}^n$ from $K$, if $y \notin K$

  - If $y_i < 0$ return “$x_i \geq 0$”, i.e., $c := -e_i, \beta := 0$
  - If $\sum_{i=1}^{n} y_i > 1$ return “$\sum_{i=1}^{n} x_i \leq 1$”, i.e., $c := (1, \ldots, 1), \beta := 1$
  - If $\sum_{i=1}^{n} y_i < 1$ return “$\sum_{i=1}^{n} x_i \geq 1$”, i.e., $c := -(1, \ldots, 1), \beta := -1$
  - If $y^T Q y > \delta$ return “$(Qy)^T x \leq \sqrt{\delta y^T Q y}$”
    It is a separating hyperplane because $y^T Q y > \delta \Rightarrow (Qy)^T y = y^T Q y > \sqrt{\delta y^T Q y}$.
    and $x^T Q x \leq \delta \Rightarrow (Qy)^T x = y^T Q x \leq \sqrt{y^T Q y \cdot x^T Q x} \leq \sqrt{y^T Q y \cdot \delta}$ using the Cauchy-Schwarz inequality.

- Unfortunately, $K$ is not full-dimensional in this case, hence we can enlarge $K$ to the following set with non-zero volume by relaxing the constraints:

$$K_\varepsilon := \left\{ x \in \mathbb{R}^n \mid 1 - \varepsilon \leq \sum_{i=1}^{n} x_i \leq 1 + \varepsilon, x_i \geq -\varepsilon, \sum_{i=1}^{n} \tilde{r}_i x_i > r - \varepsilon, x^T Q x \leq \delta + \varepsilon \right\}.$$