Optimization Methods in Finance **(EPFL, Fall 2010)**

Lecture 4: Online portfolio selection & Mean variance portfolio optimization

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Recap

Let $y^0, \ldots, y^{T-1}$ be price relatives. The return of a portfolio $x^t \in \Sigma^N$ over time horizon $[0, T]$ is

$$\prod_{t=0}^{T-1} y^t x^t$$

A best *constant-rebalanced portfolio* is a vector $x \in \Sigma^N$ attaining

$$\min_{x \in \Sigma^N} \frac{1}{T} \sum_{t=0}^{T-1} -\ln(y^t x)$$

Our goal is to prove the following theorem:

Theorem 1. *One can compute an online strategy $x_0, \ldots, x_{T-1} \in \Sigma^N$ such that*

$$\frac{1}{T} \sum_{t=0}^{T-1} \left(\ln(y^t x^*) - \ln(y^t x^t) \right) \leq 4\rho \sqrt{\frac{\ln(N)}{T}}$$

for any $x^* \in \Sigma^N$, where ρ is a bound on $\frac{y_i^t}{y_j^t}$ $\forall i, j, t$.

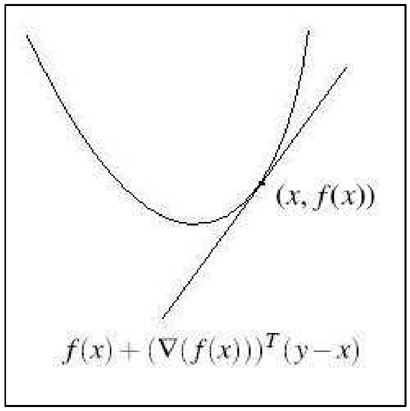
Remark: The left-hand side of the inequality is referred as *average regret*. Recall that *the first order condition of convexity* is as follows:

Lemma 2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and dom$(f) \subseteq \mathbb{R}^n$ is convex. Then,*

$$f \text{ is convex} \iff \forall x, y \in dom(f) : f(y) \geq f(x) + (\nabla f(x))^T (y - x)$$

Note that the function $f_t : \Sigma^N \to \mathbb{R}$ with $f_t(x) = -\ln(x^T y^t)$ is convex. For any $x \in \Sigma^N$, set ρ as $\|\nabla f_t(x)\|_\infty \leq \max \frac{y_i^t}{y_j^t} =: \rho \ \forall i, j, t$.

Figure 1: Illustration of Lemma 1.2



Online Portfolio Selection Using RWMA

Theorem 3. *(Reinterpretation of 1) Let $f_t : \Sigma^N \to \mathbb{R}$ be convex and differentiable for $t = 0, \dots, T-1$. One can compute $p_0, \dots, p_{T-1} \in \Sigma^N$ online such that $\forall p^* \in \Sigma^N$,*

$$\frac{1}{T}\sum_{t=0}^{T-1}[f_t(p_t) - f_t(p^*)] \le 4\rho\sqrt{\frac{\ln(N)}{T}}, \text{ where } \rho \ge \max_{t,\, x\in\Sigma^N} \|\nabla f_t(x)\|_\infty$$

Proof. To obtain such a sequence, will will apply again the randomized weighted majority algorithm. We use the following setting: The pure portfolios $e_1, \dots, e_N$ are the N experts. At time t

- $p_t \in \Sigma^N$ is the distribution on experts $\{1, \dots, N\}$ (induced by the exponential weights)
- As loss vector, we choose $\ell^t = \nabla f_t(p_t)$, where $\nabla f_t(p_t) \in [-\rho, \rho]^N$.

Recall that

$$E[\hat{L}] \le \frac{\rho\ln(N)}{\epsilon} + (1+\epsilon)L_+^j + (1-\epsilon)L_-^j, \text{ where } L_+^j = \sum_{t=0;\ell_j^t\ge 0}^{T-1}\ell_j^t,\ L_-^j = \sum_{t=0;\ell_j^t<0}^{T-1}\ell_j^t$$

In our setting,

$$\begin{aligned}
\frac{E[\hat{L}]}{T} &\le \frac{\rho\ln(N)}{\epsilon T} + (1+\epsilon)\frac{L_+^j}{T} + (1-\epsilon)\frac{L_-^j}{T} \\
&= \frac{\rho\ln(N)}{\epsilon T} + (1+\epsilon)\frac{(L_+^j + L_-^j)}{T} - \frac{2\epsilon L_-^j}{T} \\
&\le \frac{\rho\ln(N)}{\epsilon T} + (1+\epsilon)\frac{(L^j)}{T} + 2\epsilon\rho \quad (\star)
\end{aligned}$$

Here we used in ($\star$) that $\frac{L_-^j}{T} \geq -\rho$ and hence $-2\epsilon\frac{L_-^j}{T} \leq 2\epsilon\rho$. We obtain

$$\frac{E[\hat{L}] - L^j}{T} \leq \frac{\rho \ln(N)}{\epsilon T} + 3\epsilon\rho$$

We use this bound on the loss of the imaginery forecaster as follows:

$$\begin{aligned}
\frac{1}{T}\sum_{t=0}^{T-1}(f_t(p_t) - f_t(p^*)) &\overset{(\star\star)}{\leq} \frac{1}{T}\sum_{t=0}^{T-1}((\nabla f_t(p_t))^T(p_t - p^*)) \\
&= \frac{1}{T}\sum_{t=0}^{T-1}((\nabla f_t(p_t))^T p_t - (\nabla f_t(p_t))^T p^*) \\
&= \frac{E[\hat{L}]}{T} - \sum_{t=0}^{T-1}(\nabla f_t(p_t))^T p^* \\
&\leq \frac{E[\hat{L}] - L^j}{T}
\end{aligned}$$

for some j. In ($\star\star$) we used the first order condition $f(y) \geq f(x) + (\nabla f(x))^T(y-x)$ (and consequently $f(x) - f(y) \leq (\nabla f(x))^T(x-y)$). Note that

$$\frac{E[\hat{L}] - L^j}{T} \leq \frac{\rho \ln(N)}{\epsilon T} + 3\epsilon\rho \leq 4\rho\varepsilon$$

if we choose $\epsilon := \sqrt{\frac{\ln(N)}{T}}$. □

Remark: A proof of the First-Order Condition can be found e.g. in the book “Convex Optimization“. [1].

Suppose we have to solve:

$$\min_{x\in\Sigma^N} f(x) \text{ where } f:\mathbb{R}^n \to \mathbb{R} \text{ is convex and differentiable.}$$

Use the setting from before with $f_t := f \ \forall t = 0, \ldots, T-1$.

Theorem 4. *With the RWMA, one can compute an $x^* \in \Sigma^N$ such that*

$$f(x^*) - f(x) \leq \delta \textit{ for all } x \in \Sigma^N \textit{ with } T = \left(\frac{4\rho}{\delta}\right)^2 ln(N).$$

Proof. Use p_t from theorem before with $f(p_t)$ minimal. □

[1] Stephen P. Boyd, Lieven Vandenberghe: *Convex Optimization*, Cambridge University Press, p.69-70 (2004).

Mean Variance Portfolio Optimization

The following method is based on the *diversification principle* of Harry Markowitz [2]. Note that Markowitz received the Nobel Prize in economics (1990).
Suppose that N assets are available. R_i is return of asset i. $R = \sum_{i=1}^{N} R_i x_i$ is return of portfolio $x \in \Sigma^N$. Using $R = 1 + r$, (r being relative return) and $\sum_{i=1}^{N} x_i r_i$ is relative return of portfolio.

Basic notions of probability

- If x is a random variable over a finite probability space, then *expected value* of x, $E[x]$ or $\overline{x}$, is defined as $E[x] = \sum_i p_i x_i$, where p_i is the probability of x attaining the value x_i.
- *Linearity of expectation:* x, y are random variables, $\alpha, \beta \in \mathbb{R}$, then $E[\alpha x + \beta y = \alpha E[x] + \beta E[y]$.
- *Variance:* $Var(x) = E[(x - \overline{x})^2] = E[x^2] - E[x]^2$.
- *Standart deviation:* $\sigma(x) = \sqrt{\mathrm{Var}(x)}$.

  Example 5. *Rolling a dice ($x \in \{1, \ldots, 6\}$)*

  - $E[x] = 3.5$
  - $E[x^2] = (1/6)(1 + 4 + 9 + 16 + 25 + 36)$
  - $Var[x] = 2.29$
- *Covariance:* $\mathrm{Cov}(x,y) = E[(x - \overline{x})(y - \overline{y})] = E[xy] - \overline{xy}$.
- *Correlation:* $\mathrm{Corr}(x,y) = \rho(x,y) = \frac{\mathrm{Cov}(x,y)}{\sigma(x)\sigma(y)}$. Observe that $|\rho(x,y)| \leq 1$.
  - uncorrelated: $\rho(x,y) = 0$
  - positively correlated: $\rho(x,y) > 0$
  - negatively correlated: $\rho(x,y) < 0$
- *Variance of sum:* Let $x_1, \ldots, x_n$ be random variables. Then

$$\begin{aligned}
\mathrm{Var}\Big[\sum_{i=1}^{n} x_i\Big] &= E\Big[\sum_{i=1}^{n}(x_i - \overline{x_i})\Big]^2 \\
&= E\Big[\sum_{i,j}(x_i - \overline{x_i})(x_j - \overline{x_j})\Big] \\
&= E\Big[\sum_{i,j} x_i x_j - x_i \overline{x_j} - \overline{x_i} x_j + \overline{x_i}\,\overline{x_j}\Big] \\
&= \sum_{i,j} E[x_i x_j] - \overline{x_i}\,\overline{x_j} \\
&= \sum_{i,j} \mathrm{Cov}(x_i, x_j)
\end{aligned}$$

[2] Markowitz, H., 1952. Portfolio selection. Journal of Finance 7, p.77-91.