COMBINATORIAL OPTIMIZATION
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Fall 2011
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Sample exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ
max points:	8	10	8	6	8	8	8	48
achieved points:								

Check whether the exam is complete: it should have 7 pages (Exercises 1–7). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, additional paper can be asked from the exam supervision.

Use neither pencil nor red colored pen!

Duration: 180 min

Exercise 1: (Multiple Choice, points $\{-1,0,1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!**
Total number of points achieved cannot be negative.

a) Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a polyhedron with $A \in \mathbb{R}^{m\times n}, b \in R^m$ and $A'x \leq b'$ a subsystem of $Ax \leq b$. Then $F = \{x \in \mathbb{R}^n \colon A'x = b'\}$ is a face of P.	○ yes	○ no
b) If $Ax \leq b$ is TDI, then the system $Ax \leq b, ax \leq \beta$ is also TDI.	○ yes	○ no
c) In an undirected graph, the maximum cardinality of a matching equals the minimum cardinality of a vertex cover.	○ yes	○ no
d) The rank function of a matroid is submodular.	○ yes	○ no
e) Let $E \neq \emptyset$ be a finite set and let $\mathscr{I} = 2^E$ be its power set. Then $(E, \mathscr{I})$ is a matroid.	○ yes	○ no
f) Let $P \subseteq \mathbb{R}^n$ be an integral polyhedron with non-empty interior. Then $\mathrm{vol}(P) \geq \frac{1}{n!}$.	○ yes	○ no
g) A digraph contains an arborescence if and only if there exists a node from which all other nodes are reachable.	○ yes	○ no
h) Finding a maximum weight matching in a graph is in NP.	○ yes	○ no

Exercise 2: (10 points)

The problem WEIGHTED EDGE COVER is defined as follows: Given a graph $G = (V, E)$, a weight function on the edges $c : E \to \mathbb{R}^+$, find a minimum weight set $S \subseteq E$ of edges such that every vertex $v \in V$ is an endpoint of an edge in S.

1. Give an algorithm that solves the uniform problem, i.e. $c \equiv 1$, in polynomial time.
2. Show that the following algorithm is not optimal for the general problem:
   Compute a minimum weight maximum matching M in G. For every vertex v not covered by M, add the edge with least cost adjacent to v to M. Return M.
3. Give an algorithm that solves the general problem in polynomial time.
   Hint: Use the following construction: create a copy G' of G. Set all edge weights to zero in G'. Let $\hat{G}$ be the disjoint union of G' and G. Add an edge to $\hat{G}$ between a vertex v from G and its copy v' from G' and set its weight to the minimum weight of all incident edges at v in G.
   Continue using matching techniques in $\hat{G}$.

Solution:

Use reverse side if you need more space

Exercise 3: (8 points)
Show the following:
A face F of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

Solution:

Use reverse side if you need more space

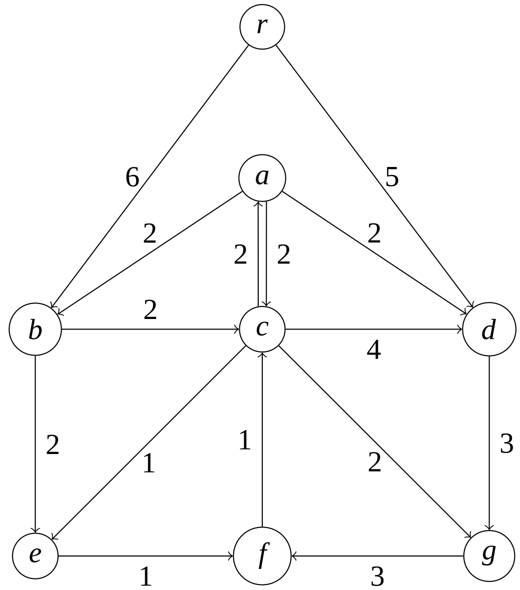
Exercise 4: (6 points)
Let $G = (V, E)$ be a (undirected) graph.

Suppose that we have an oracle that decides whether a given graph contains a perfect matching.
Design an algorithm that finds a maximum matching in G using $O(|V| + |E|)$ calls to the oracle.

Solution:

Use reverse side if you need more space

Exercise 5: (8 points)
Compute a minimum weight arborescence rooted at r in the following digraph.



Prove the optimality of your solution!

Solution:

Use reverse side if you need more space

Exercise 6: (8 points)
The problem 2-SAT is defined as follows:
Given a set X of variables and a collection $\mathscr{Z}$ of clauses over X, each containing 2 literals. Is $\mathscr{Z}$ satisfiable?

Show that 2-SAT can be solved in polynomial time.

Hint: Consider the following digraph. The nodes are the literals over X and there an arc (λ_1, λ_2) if and only if the clause $\{\overline{\lambda}_1, \lambda_2\}$ is a member of $\mathscr{Z}$.

Solution:

Use reverse side if you need more space

Exercise 7: (8 points)
Consider the following computational problems on undirected graphs:

$$\text{Independent-Set} := \{\langle G,k\rangle \mid G \text{ contains a stable set of size } k\}$$
$$\text{Clique} := \{\langle G,k\rangle \mid G \text{ contains a clique of size } k\}$$
$$\text{Half-Clique} := \{\langle G\rangle \mid G=(V,E) \text{ contains a clique of size } \frac{|V|}{2}, |V| \text{ is even}\}$$

Prove that Independent-Set polynomially transforms to Clique, and Clique polynomially transforms to Half-Clique.

Solution:

Use reverse side if you need more space