École Polytechnique Fédérale de Lausanne
Eisenbrand, Moldenhauer

November 17, 2011

Randomized Algorithms (Fall 2011)

Assignment 5

Due date: 10:00am, November 29, 2011 — 25+12 points

Problem 1 (Clarkson I) — 3 points

Show how to use the algorithm Clarkson I to obtain a linear time algorithm (in the number of constraints) for fixed dimension.

Problem 2 (Clarkson II) — 0+2 points

Show that an iteration in the Clarkson II algorithm presented in class is successful, *i.e.*, $\mu(V_R) \leq \frac{1}{3d}\mu(H)$ with probability at least $\frac{1}{2}$.

Problem 3 (Seidel LP) — 3 points

Show, that there is a constant c such that the runtime of the SeidelLP algorithm for n constraints in d dimensions is $T(n,d) \leq cnd!$.

Problem 4 (Birthday Paradox) — 3 points

Personal Identification Numbers (PIN) in Switzerland have 6 digits. Assume that a bank issues PINs uniformly at random with replacement. Calculate the maximum number of clients a bank can have such that the probability that two clients have the same PIN is at most 1/2.

Problem 5 (Inequalities) — 4 points

Show the following inequalities:

(a) $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$.

(b) $e^x(1-x^2) \leq 1+x$ for $|x| \leq 1$.

Problem 6 (Poisson random variables) — 12 points

A Poisson random variable X with parameter μ is given by the following distribution

$$\Pr(X = j) = \frac{e^{-\mu}\mu^j}{j!} \ .$$

Note that μ is the expected value of X.
Let X and Y be independent Possion random variables with expected values μ and ν, respectively.

(a) Show that the sum of X and Y is a Poisson random variable with expected value $\mu + \nu$.

(b) Show the following Chernoff type bound.
If $x > \mu$, then

$$\Pr(X \geq x) \leq \frac{e^{-\mu}(e\mu)^x}{x^x} \ .$$

If $x < \mu$, then

$$\Pr(X \leq x) \leq \frac{e^{-\mu}(e\mu)^x}{x^x} \ .$$

(c) Show that if μ is an integer and $\mu \geq 1$ we have

$$\Pr(X \geq \mu) \geq \frac{1}{2}\,.$$

Hint: Show first that $\Pr(X = \mu + h) \geq \Pr(X = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

(d) Show under the same assumptions that also

$$\Pr(X \leq \mu) \geq \frac{1}{2}\,.$$

Hint: Show that $\Pr(X = \mu - h) \geq \Pr(X = \mu + h + 1)$ for $0 \leq h \leq \mu$. Then determine a lower bound on $\Pr(X = \mu - h) - \Pr(X = \mu + h + 1)$ and an upper bound on $\Pr(X \geq 2\mu + 2)$.

Problem 7 (Repeated balls into bins) 0+5 points

Consider a process that uses multiple rounds. Every round, balls are thrown independently and uniformly at random into n bins. Any ball that ends up in a bin by itself is removed from consideration. The remaining balls are thrown again in the next round. The process begins with n balls and ends when every ball has been removed.

(a) If there are b balls at the start of a round, what is the expected number of balls at the start of the next round?

(b) Suppose that every round the number of removed balls was exactly the expected number of removed balls. Show that all the balls would be removed in $O(\log \log n)$ rounds.

Problem 8 (Maximal Sequences) 0+5 points

Consider k distinct values chosen uniformly at random from the set $\{0, \ldots, n-1\}$. We consider maximal sequences of values that are obtained by the choice of the k values (*e.g.*, if we choose values $1, 5, 2, 8, 7$ the maximal sequences are $\{1, 2\}, \{5\}, \{7, 8\}$). Let X be the number of maximal sequences obtained. Compute the expected value of X.