École Polytechnique Fédérale de Lausanne
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November 3, 2011

Randomized Algorithms (Fall 2011)

Assignment 4

Due date: 10:00am, November 15, 2011 — 25 points

Problem 1 (Weighted Majority) — 4 points

Consider the weighted majority algorithm for N experts and loss vectors $\ell^t \in [-\rho, \rho]^N$. Show that the algorithm produces an expected loss of at most

$$\mathbb{E}[L] \leq \frac{\rho \ln(N)}{\epsilon} + (1+\epsilon) \sum_{t: \ell_j^t \geq 0} \ell_j^t + (1-\epsilon) \sum_{t: \ell_j^t < 0} \ell_j^t \ .$$

Problem 2 (Weighted Majority for LPs) — 4 points

Consider the problem Set Cover: given a ground set U and m subsets $S_i \subseteq U$, find the smallest number of sets that cover U.
Consider the following LP relaxation for this problem

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} x_i \\ \text{s.t.} \quad & \sum_{i: e \in S_i} x_i \geq 1 \qquad \forall e \in U \\ & x_i \geq 0 \qquad 1 \leq i \leq m \ . \end{aligned}$$

Use the weighted majority algorithm to find a feasible solution x to this LP with $1^{\mathrm{T}} \cdot x \leq (1+\epsilon)^{\mathrm{T}} \cdot x^*$, where x^* is the LP optimum. Show that your algorithm runs in time polynomial in $1/\epsilon$ and the size of the largest set of the instance.

Problem 3 (Clarkson) — 8 points

Consider the following problem: Given a set $H \subseteq \mathbb{R}^d$ with $|H| = m$. Find a *smallest enclosing ball* of H. Recall that the ball with radius R and center $c \in \mathbb{R}^d$ is the set $B_{R,c} = \{x \in \mathbb{R}^d \colon \|x - c\| \leq R\}$. We denote the smallest enclosing ball of a subset $G \subseteq H$ by $b^*(G)$.

(a) Let $G \subseteq H$. Show that $B_{R,c} = b^*(G)$ if and only $B_{R,c}$ contains G and c lies in the convex hull of the points $\{g \in G \colon \|g - c\| = R\}$.
Hint: Use the separation theorem for convex sets.

(b) Conclude that for any $G \subseteq H$, there exists a *basis* $B \subseteq G$ with $|B| \leq d+1$ and $b^*(B) = b^*(G)$.
Hint: Use Caratheodory's theorem.

(c) Prove the following lemma:
Let G and H (multi-)sets of points in $\mathbb{R}^d$ with $|H| = m$ and let $1 \leq r \leq m$. Then for random $R \in \binom{H}{r}$:

$$E[|V_R|] \leq (d+1)(m-r)/(r+1),$$

where $V_R = \{h \in H \mid b^*(G \cup R) \text{ does not contain } h\}$.

(d) Formulate and analyze a Clarkson 1 algorithm for smallest enclosing ball.

Problem 4 (Degeneracy of LPs) 5 points

Consider the following general minimization LP for $x \in \mathbb{R}^n$:

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ \text{s.t.} \quad & Ax \le b \,. \end{aligned}$$

Assume that the polyhedron is non-empty and bounded. Show that we can then make the following assumptions without loss of generality.

(a) The objective function is $c = (1, 0, \ldots, 0)$.

(b) The minimum point is unique, *i.e.*, a vertex of the polytope.

(c) Each vertex of the polytope is defined by exactly n constraints.
(*Hint*: perturb b by adding ϵ^i to the ith component of b.)

Problem 5 (Lexicographical Maximum) 4 points

Let $P = \{x \in \mathbb{R}^n : Ax \le b, -M \le x \le M\}$. Show that the problem of finding a lexicographically maximal point of P is a linear programming problem.
Hint: Argue that we can assume that all entries in A and b are integer and that the lexicographical maximum is attained in a vertex of the polytope. Then, given two vertices x_1 and x_2 of the polytope, give a lower bound on the distance of their components $|x_1^i - x_2^i|$ (use Cramer's rule and Hadamard's inequality). Use this lower bound and M to construct an objective function. Show that the minimization w.r.t. this objective function yields the lexicographically maximal point.