École Polytechnique Fédérale de Lausanne
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Randomized Algorithms (Fall 2011)

Assignment 3

Due date: 10:00am, November 01, 2011 — 25 points

Problem 1 (Minimum Congestion I) — 4 points

Give an approximation algorithm for the following problem with approximation factor of $O(\frac{\log n}{\log\log n})$.
Min-Max Congestion: Given a connected undirected graph $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$ pairs of vertices, compute (s_i, t_i) paths such that the maximum congestion on an edge is minimized.

Problem 2 (Minimum Congestion II) — 4 points

Assume that the Min-Max Congestion is lower bounded by $C = \Omega(\log n)$. Show that under this assumption, the Min-Max Congestion problem can be approximated with a constant factor.

Problem 3 (Quadratic Programs via Semidefinite Programming) — 5 points

Consider the following quadratic program:

$$\begin{aligned} \max \quad & \sum_{1\le i,j,\le n} a_{ij}x_ix_j \\ \text{s.t.} \quad & x_i \in \{-1,+1\} \qquad i = 1,\ldots,n \end{aligned}$$

where the matrix $A = (a_{ij})_{i,j}$ is positive semidefinite. Show how this program can be approximated with a factor of $\frac{2}{\pi}$.

Hint: Relax the program to a vector program and round its solution. Then, show that $\mathbb{E}[x_ix_j] = \frac{2}{\pi}\arcsin(v_i \cdot v_j)$, where x_i and x_j are the integral values of the rounded solution. Use the following facts to yield the final result.
If $A \succeq 0$ and $B \succeq 0$, then $\sum_{i,j} a_{ij}b_{ij} \ge 0$.
If $X \succeq 0$, $|x_{ij}| \le 1$ for all i, j, and $Z = (z_{ij})$ such that $z_{ij} = \arcsin(x_{ij}) - x_{ij}$, then $Z \succeq 0$.

Problem 4 (Inequalities) — 4 points

Show the following inequalities for $0 \le \varepsilon \le 1/2$:

1. $(1-\varepsilon)^x \le (1-\varepsilon x)$ for $x \in [0, 1]$.
2. $(1+\varepsilon)^{-x} \le (1-\varepsilon x)$ for $x \in [-1, 0]$.
3. $\ln\left(\frac{1}{1-\varepsilon}\right) \le \varepsilon + \varepsilon^2$.
4. $\ln(1+\varepsilon) \ge \varepsilon - \varepsilon^2$.

Problem 5 (Weighted Majority I) — 4 points

Consider the randomized weighted majority algorithm and suppose that the loss-vectors at time t satisfy $\ell^t \in [0, \rho]^N$ for $t = 0, \ldots, T$. Show that the expected loss of the forecaster is bounded by

$$E[L] \le \frac{\rho \cdot \ln N}{\varepsilon} + (1+\varepsilon) \cdot L^j,$$

if one uses the update rule $w_j := w_j(1-\epsilon)^{\ell_j^T/\rho}$. The loss accumulated by expert j is $L^j = \sum_{t=0}^{T} \ell_j^t$.

Problem 6 (Weighted Majority II) 4 points

Suppose you have some initial belief about the quality of the experts. This belief is represented by a probability distribution on the experts p_j, $j = 1, \ldots, N$ with $p_j > 0$ and $\sum_{j=1}^{n} p_j = 1$. We modify the weighted majority algorithm by setting the initial weights $w_j := p_j$. Show that this modification results in a guarantee

$$E[L] \leq \frac{\ln(1/p_j)}{\varepsilon} + (1+\varepsilon) \cdot L^j.$$

Suppose now that we have a countably infinite number of experts. Use the result above to argue that one can guarantee

$$E[L] \leq \frac{2 \cdot \ln(j) + 10}{\varepsilon} + (1+\varepsilon) \cdot L^j.$$

by choosing a suitable probability distribution on the experts.