CONVEXITY
PROF. F. EISENBRAND

Fall 2015
2.11.2015

Practice Exam

Last name: First name:

Exercise:	1	2	3	4	Σ
max points:	10	10	10	10	40
achieved points:					
chosen exercises:					

Remark - read me first:
The exercises in this practice exam are intended to give you an idea about how the final exam will look like. These exercises only cover a subset of the topics of the class. We advise you to also refresh your memory on all the other subjects. Good luck with your preparations! The exam will likely have 7 exercises from which you will choose 6, and its length will be 3 hours.

Duration: 1.5 hours

Grading: You can choose to work on 3 of the exercises.
Indicate your choice above!

Before you start:

- Check whether the exam is complete: It should have 5 pages.
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1: (10 points)
Let $K \subseteq \mathbb{R}^n$ be a set. Define the convex hull of K.

1. Prove that if $A \subset B \subseteq \mathbb{R}^n$ then $conv(A) \subset conv(B)$.
2. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Prove that the set $X = \{x \in \mathbb{R}^n : Ax \leq b\}$ is convex.
3. Let $f : \mathbb{R}^d \to \mathbb{R}^k$ be an affine map.
   (a) Prove that if $C \subseteq \mathbb{R}^d$ is convex, then $f(C)$ is convex as well. Is the preimage of a convex set always convex?
   (b) For $X \subseteq \mathbb{R}^d$ arbitrary, prove that $conv(f(X)) = f(conv(X))$

Solution:

Use reverse side if you need more space

Exercise 2 (10 points)

1. Let $K \subseteq R^d$ be closed, and convex set that is centrally symmetric about the origin.
   (a) State the Minkowski Theorem for general lattices.
   (b) Give examples to show that none of the assumptions on K can be omitted. That is, find K that is closed and bounded, but not centrally symmetric, for which Minkowski's theorem does not hold. Repeat again while in turns omitting "closed" and "convex".
2. Let $\Lambda \subset R^d$ be a general lattice. Show that there must exist a non-zero lattice point $x \in \Lambda \backslash \{0\}$ with
   $$||x||_\infty \leq (\det(\Lambda))^{\frac{1}{d}}$$
   and show that this bound cannot be improved in general.
3. Let $\Lambda = \Lambda(A)$ where
   $$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$
   Show there exist a nonzero lattice point x with
   (a) $||x||_\infty \leq \frac{3}{2}$
   (b) $||x||_2 \leq 2$

Solution:

Use reverse side if you need more space

Exercise 3 (10 points)

1. Define an ellipsoid $E = E(A, b) \subset \mathbb{R}^n$ for $A \in \mathbb{R}^{d\times d}$ a nonsingular matrix, and $b \in \mathbb{R}^d$. Write down a relation between the volume of $E = E(A, b)$ and v_n, the volume of the unit ball B_1^n in terms of A.

2. For a nonempty convex body $K \subseteq \mathbb{R}^d$ define the *width* of K by

$$w(K) = \min_{u\in\mathbb{Z}^d\setminus\{0\}} \max_{x,y\in K} u^\intercal(x - y).$$

   a) Let $E = E(A, b) \subset \mathbb{R}^n$ be an ellipsoid for some non-singular matrix $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Suppose we want to find $u \in \mathbb{Z}^d \setminus \{0\}$ for which

$$\max_{x,y\in E} u^\intercal(x - y) = w(E).$$

   Show that the problem of finding such a u can be transformed into finding a shortest vector in a lattice that you should specify.

   b) Give bounds for $w(K)$ in terms of $w(E_{in})$, the maximum volume ellipsoid contained in K and $w(E_{out})$.

3. For a vector of positive reals $a = (a_1, a_2, \ldots, a_n) \in R^n$ let $K(a) \subseteq \mathbb{R}^n$ be the parallelepiped

$$K(a) = \{x \in \mathbb{R}^n : -a_i \le x_i \le a_i,\ \forall i = 1, \ldots, n\}.$$

   c) Using the fact that B_1^n is a maximum volume ellipsoid of $K((1, 1, \ldots, 1))$, or otherwise, find a maximum volume ellipsoid $E_{in}(a)$ of $K(a)$ for $a = (a_1, \ldots, a_n)$ a vector of positive reals.

   d) By what factor do we need to expand $E_{in}(a)$ around it's center to obtain an ellipsoid that contains $K(a)$ - i.e. what is minimal value of c s.t. $c(E_{in}(a) - b) + b \supseteq K(a)$?

Solution:

Use reverse side if you need more space

Exercise 4 (10 points)
Let $A, B \subseteq \mathbb{R}^n$ be nonempty compact bodies.

1. Define the *Minkowski sum* $A+B$ of the sets A and B. State the *Brunn-Minkowski inequality* for A and B.

2. If $A \subset \mathbb{R}^n$ is convex, compact and nonempty, show that $A + A = 2A$. For each $n \in \mathbb{N}$ and positive reals c_1, c_2 exhibit $A, B \subset \mathbb{R}^n$ such that $vol(A) = c_1^n, vol(B) = c_2^n$ and for which an equality is attained in the Brunn-Minkowski inequality.

3. Prove the following inequality: Let x, y be positive reals and let $t \in (0, 1)$. Show that

$$tx + (1-t)y \geq x^t y^{1-t}$$

4. Use the above results to show that for any nonempty compact sets $C, D \subset \mathbb{R}^n$ and every $\lambda \in (0, 1)$ it holds that

$$vol(\lambda C + (1-\lambda)D) \geq (vol(C))^{\lambda} (vol(D))^{1-\lambda}$$

Solution:

Use reverse side if you need more space