Scheduling with AND/OR precedence constraints

Network Given a set of jobs $V$ (AND-nodes) and waiting conditions $W$ (OR-nodes). They form a directed bipartite graph $B$ with the arc set $A$.

Processing times/time lags integral weights $-M < d_{pq} < M$ for $(p, q) \in A$.

Starting condition $S_j \geq 0$ for all $j \in V$.

Problem formulation

\[
\begin{align*}
S_j & \geq \max_{(w, j) \in A} (S_w + d_{wj}) \quad \text{for } j \in V \\
S_w & \geq \min_{(j, w) \in A} (S_j + d_{wj}) \quad \text{for } w \in W
\end{align*}
\]

AND OR

Tropical Polyhedra

Tropical numbers $\mathbb{T} = \mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$ each with $\infty$.

Tropical operations $\oplus = \min$ and $\otimes = +$.

Remark: There is no additive inverse.

Inequality systems Systems of the form $A \otimes x \leq B \otimes x$ define tropical polyhedra where $A \otimes x = (\oplus_{i \in I} a_i \otimes x_i)$.

Theorem (Möhring, Skutella, Stork)

The following problems are polynomial time equivalent and belong to \text{NP} \cap \text{co-NP}:

- Finding a minimal schedule in a network with and/or-precedence constraints.
- Checking the feasibility of a tropical polyhedron.

- No polynomial time algorithm known
- Feasibility is equivalent to tropical linear programming, bisection or homogenization approach

Theorem (J., L. 2015)

The set of feasible schedules is described by those bipartite subgraphs of $B$ in which every node in $W$ has at least one in-going arc. These subgraphs are directed covector graphs.

Tropical linear programming - a graph algorithm

Theorem (Allamigeon, Benchimol, Gaubert, J.)

For every generic instance of the tropical simplex method there is a classical analogue. Both are polynomial-time equivalent.

- New approach with promising complexity
- Relation between classical simplex method for arbitrary polyhedra and shortest path algorithms
- Genericity no restriction via symbolic perturbation

The orthogonal projection of the set of feasible schedules onto the coordinates in $V$ is the tropical polyhedron given by

\[
\min_{(w, j) \in A} (S_j - d_{wj}) \geq \min_{(j, w) \in A} (S_j + d_{wj}) \quad \forall w \in W.
\]

Weighted digraph polyhedron for $B$

All the points $(y, z) \in \mathbb{R}^{V \cup W}$ with $y_j \leq z_w \leq d_{wj}$ and $y_j \geq z_w \leq d_{wj}$ for all arcs.

Tropical covector graphs

A bipartite graph $G$ on $V \sqcup W$ is a covector graph for a weight matrix $D \in \{\mathbb{R} \cup \{\infty\}\}^{V \times W}$ if and only if the following are satisfied:

- Minimality: for every pair of subsets $P \subseteq V$ and $Q \subseteq W$ with $|P| = |Q|$, every perfect matching of $G$ restricted to $P \sqcup Q$ is a minimal matching of the complete bipartite graph $P \times Q$ with the weights given by the corresponding submatrix of $D$.

Completeness: if there are more minimal perfect matchings in $P \times Q$ then each of them is contained in $G$.

- Tropical oriented matroid
- Similar to graphs in the hungarian method
- Shortest path in bipartite graph

YES $\rightarrow$ NO $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$

References

- Allamigeon, Xavier and Benchimol, Pascal and Gaubert, Stéphane and Joswig, Michael. Combinatorial simplex algorithms can solve mean payoff games, SIAM J. Opt. 24 (2014) 4

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