Prof. Friedrich Eisenbrand
Nicolai Hähnle

Discussions from: October 11, 2010

Combinatorial Optimization

Fall 2010

Assignment Sheet 2

Exercise 2

Recall the inequalities ($\star$) for a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Z}_+$ and maximum weight μ of any matching:

$$\begin{aligned} \sum_{v \in V} y_v + \sum_{U \subseteq V, |U| \text{ odd}} \frac{|U|-1}{2} \cdot y_U &\leq \mu & \\ y_u + y_v + \sum_{U \subseteq V, |U| \text{ odd}, uv \in E(U)} y_U &\geq w_{uv} & \forall\, e = uv \in E \\ y_v &\in \mathbb{Z}_+ & \forall\, v \in V \\ y_U &\in \mathbb{Z}_+ & \forall\, U \subseteq V, |U| \text{ odd} \end{aligned}$$

Suppose that $G = (V, E)$ and $w \in \mathbb{Z}_+^E$ is a counter-example to the feasibility of these inequalities with $|V| + |E|$ minimal. Assume that G is not connected, for the purpose of obtaining a contradiction.

If G is not connected, then we can write G as the disjoint union of two parts $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with no connection between G_1 and G_2. Note that it is irrelevant whether G_1 and G_2 are themselves connected. The important fact is that $|V_1| + |E_1| < |V| + |E|$, and so there exists an integral vector y' satisfying the inequalities ($\star$) for G_1. Similarly, there exists a vector y'' satisfying the inequalities for G_2. Let us define the vector y by simply combining v' and v'' in the obvious way. Formally:

$$y_v := \begin{cases} y'_v & \text{if } v \in V_1 \\ y''_v & \text{if } v \in V_2 \end{cases} \qquad \forall\, v \in V$$

$$y_U := \begin{cases} y'_U & \text{if } U \subseteq V_1 \\ y''_U & \text{if } U \subseteq V_2 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, U \subseteq V, |U| \text{ odd}$$

Does the vector y satisfy the inequalities ($\star$)? It certainly is integral. It also satisfies

$$y_u + y_v + \sum_{U \subseteq V, |U| \text{ odd}, uv \in E(U)} y_U \geq w_{uv}$$

for every edge $e = uv \in E$, because every edge is either in G_1 or in G_2, and so the fact that the corresponding inequality holds for y' or y'', respectively, shows that it holds for y. Finally, we have

$$\begin{aligned}
& \sum_{v\in V} y_v + \sum_{U\subseteq V, |U| \text{ odd}} \frac{|U|-1}{2} \cdot y_U \\
= & \left(\sum_{v\in V_1} y'_v + \sum_{U\subseteq V_1, |U| \text{ odd}} \frac{|U|-1}{2} \cdot y'_U \right) + \left(\sum_{v\in V_2} y''_v + \sum_{U\subseteq V_2, |U| \text{ odd}} \frac{|U|-1}{2} \cdot y''_U \right) \\
\leq & \mu_1 + \mu_2
\end{aligned}$$

But clearly $\mu_1 + \mu_2 \leq \mu$, because we can simply take the union of maximal matchings from G_1 and G_2 to obtain a matching of G. This completes the proof that y is a feasible solution for $(\star)$ for G, which contradicts the choice of G, w as a counter-example. So a minimal counter-example must be connected.

Exercise 3

Clearly $\mu_{w'} \geq \mu_w - \lfloor |V|/2 \rfloor$: take any maximum weight matching M of G with respect to w; then by the conditions of the exercise, $|M| = \lfloor |V|/2 \rfloor$ and so the weight of this matching with respect to w' is exactly $\mu_w - \lfloor |V|/2 \rfloor$. The converse follows if we can show that there exists a maximum weight matching with respect to w' of size $\lfloor |V|/2 \rfloor$.

So let us take a maximum weight matching M' with respect to w' with $|M'| < \lfloor |V|/2 \rfloor$. Our goal is to find a larger matching of at least the same weight. Then by iteration, we can find a maximum weight matching of size $\lfloor |V|/2 \rfloor$. To that end, take an arbitrary maximum weight matching M with respect to w and consider the symmetric difference $M \Delta M'$.

Every vertex is adjacent to at most two edges of $M \Delta M'$, so $M \Delta M'$ is the union of vertex-disjoint paths and cycles. Furthermore, edges from M and edges from M' are alternating, and so since $|M| > |M'|$, there has to be a path P of odd length in $M \Delta M'$ such that $|P \cap M| = |P \cap M'| + 1$.

Now consider the matching $M \Delta P$ that we get by “flipping” the edges in the path P from one matching to the other. Clearly $|M \Delta P| = |M| - 1$ and by the postulate of the problem statement, this implies

$$w(M \Delta P) < w(M)$$

We compute:

$$w(M \Delta P) = w(M) - w(P \cap M) + w(P \cap M')$$

and therefore:

$$w(P \cap M) > w(P \cap M')$$

Keep in mind that all weights are integral and continue:

$$\begin{aligned}
w'(P \cap M) &= w(P \cap M) - |P \cap M| \\
&\geq w(P \cap M') + 1 - |P \cap M| \\
&= w(P \cap M') - |P \cap M'| = w'(P \cap M')
\end{aligned}$$

So we can improve the matching M' by flipping edges of the path to obtain the matching $M' \Delta P$ that has one more edge and satisfies

$$w'(M' \Delta P) = w'(M) - w'(P \cap M') + w'(P \cap M) \geq w'(M)$$

Exercise 4

6. We will use the linear algebra method to construct an example set cover instance with a fractional optimum very close to 2, and an integer optimum close to $\log m$, where m is the number of points.

   As set of points we are going to use the set of non-zero vectors of a k-dimensional vector space over the field $\mathbb{F}_2$:

   $$V := \mathbb{F}_2^k \setminus \{0\}$$

   The sets will also be indexed by non-zero vectors from $\mathbb{F}_2^k$. For every $u \in \mathbb{F}_2^k \setminus \{0\}$, we define a set

   $$S_u := \{u \in V \mid u^T v = 1\}$$

   Note that the scalar product $u^T v$ is computed in $\mathbb{F}_2$, so the computation is done modulo 2. The points in S_u form a $(k-1)$-dimensional affine subspace, and so

   $$|S_u| = 2^{k-1}$$

   Since the construction has a nice symmetry,[1] every point is contained in exactly 2^{k-1} sets. In other words, we obtain a feasible fractional solution by setting all variables to the same value:

   $$x_u = \frac{1}{2^{k-1}}$$

   The objective function value of this solution is

   $$\sum_{u \in \mathbb{F}_2^k \setminus \{0\}} x_u = \frac{2^k - 1}{2^{k-1}} = 2 - \frac{1}{2^{k-1}}$$

   Let us now prove that the optimal integer solution uses exactly k sets. Let $e_1, \ldots, e_k \in \mathbb{F}_2^k$ be the unit vectors, then the sets $S_{e_1}, \ldots, S_{e_k}$ cover all points of V. Simply observe that every $v \in V$ has at least one non-zero entry, say in the j-th coordinate. Then $e_j^T v = 1$, and therefore $v \in S_{e_j}$.

   Now suppose we have sets $S_{u_1}, \ldots, S_{u_r}$ with $r < k$. We claim that there is at least one point in V that is not covered by these sets. To see this, consider the system of linear equations:

   $$\begin{aligned} u_1^T x &= 0 \\ &\vdots \\ u_r^T x &= 0 \end{aligned}$$

[1] This can be stated formally: if we use the same ordering of vectors for both points and sets, then the coefficient matrix A of the linear program is symmetric. This is just a side effect of the fact that the scalar product is symmetric.

Since $r < k$, the set of solutions is a subspace of dimension at least 1. Therefore, there exists a non-zero solution $v \in V$. Furthermore, $u_j^T v = 0$ for all j implies $v \notin S_{u_j}$ for all j, which means that v is not covered. Keeping in mind that $m = |V| = 2^k - 1$, we conclude that

$$\frac{OPT}{OPT_f} = \frac{k}{2 - \frac{1}{2^{k-1}}} \geq \frac{k}{2} = \Omega(\log m)$$

Remark: If you think about the above system of linear equations a little, you will soon realize that *every* basis of $\mathbb{F}_2^k$ corresponds to an optimal solution of the set cover instance. So the number of optimal solutions is huge, and they are very evenly distributed. This is not an accident: the linear algebra method is typically used for problems in combinatorics where the number of "optimal" or "extremal" configurations is very large.