Prof. Friedrich Eisenbrand
Nicolai Hähnle

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Combinatorial Optimization

Fall 2010

Assignment Sheet 1

Exercise 2

It is satisfied with equality by each $\mathbf{x} \in \mathbf{F}$. Let $x \in F$, then $A'x = b'$. So:

$$c^T x = 1^T A' x = 1^T b' = \delta$$

(Note that there was a missing transpose on the assignment sheet; it should be $c^T = 1^T A'$ in the definition of c.)

and consequently $\mathbf{c^T x'} < \delta$. We have $x' \in P \setminus F$. Let us denote the inequalities of $A'x \le b'$ as $a_j'^T x \le b_j$ for $j = 1 \dots k$, and rearrange the inequalities such that $a_1'^T x' < b_1'$.

$$c^T x' = 1^T A' x' = \sum_{j=1}^{k} a_j'^T x' = \underbrace{a_1'^T x'}_{< b_1'} + \sum_{j=2}^{k} \underbrace{a_j'^T x'}_{\le b_j'} < \sum_{j=1}^{k} b_j' = 1^T b = \delta$$

and so $c^T x' < \delta$.

One has $\mathbf{F} = \{\mathbf{x} \in \mathbf{P} \colon \mathbf{A'x} = \mathbf{b'}\}$. Let us show $F \subseteq \{x \in P \colon A'x = b'\}$. Let $x \in F$. Then by definition of F, we have $x \in P$, and by definition of P, we have $A'x \le b'$. We only need to check $A'x = b'$. Assume that this is not true, then without loss of generality we have $a_1'^T x < b_1'$. But then the same computation as above shows that $c^T x < \delta$, which is a contradiction.

Now let us show $\{x \in P \colon A'x = b'\} \subseteq F$. Let $x \in P$, $A'x = b'$. Then

$$c^T x = 1^T A' x = 1^T b' = \delta$$

and therefore $x \in F$.

Exercise 4 ($\star$)

Show the following: A face F of $P = \{x \in \mathbb{R}^n \colon Ax \le b\}$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$.

Let F be an inclusion-wise minimal face. Write

$$F = \{x \in P \colon A'x = b'\}$$

where $A'x \le b'$ is the maximal possible subsystem of $Ax \le b$ with that property, and let

$$G = \{x \in \mathbb{R}^n : A'x = b'\}$$

Assume $F \neq G$, then there is a point $z \in G \setminus F$. In particular, $z \notin P$. Furthermore, there exists a point $y \in F$. Consider the line segment parameterized by:

$$w(t) = (1-t)y + tz, t \in [0,1]$$

Let $a^T x \le \beta$ be the first inequality of $Ax \le b$ that is violated as $w(t)$ moves from y to z, and let $t \in [0,1)$ such that $a^T w(t) = \beta$. Then

$$F' = \{x \in P : A'x = b', a^T x = \beta\}$$

is a face of P by Exercise 2, it is clearly contained in F, and it is non-empty because $w(t) \in F'$. Finally, note that $a^T x = \beta$ cannot be contained in the system $A'x = b'$, because $a^T w(t) = \beta$ does not hold for all $t \in [0,1]$. Therefore, F' is defined by a subsystem of equations that is strictly bigger than any subsystem that defines F (remember that we chose $A'x = b'$ to be maximal!) and so $F' \neq F$. In conclusion, F' is a proper sub-face of F, which contradicts the inclusion-wise minimality of F. So the assumption was wrong, in fact we have $F = G$.

Let F be a face of P such that $F = \{x \in \mathbb{R}^n : A'x = b'\}$ for a subsystem $A'x \le b'$ of $Ax \le b$. Assume that F is not inclusion-wise minimal, i.e. there is a proper sub-face $F' \subsetneq F$. We can write

$$F' = \{x \in P : A'x = b', A''x = b''\}$$

for a subsystem $A''x \le b''$ of $Ax \le b$. Let $y \in F'$ and $z \in F \setminus F'$. Then the line through y and z is contained entirely in F, however there will be one inequality $a^T x \le \beta$ of the system $A''x \le b''$ that is not parallel to the line through y and z. This means that the line cannot be entirely contained in P. This is a contradiction, and so F must be inclusion-wise minimal.