Exercise 2

It is satisfied with equality by each \( x \in F \). Let \( x \in F \), then \( A'x = b' \). So:

\[
c^T x = 1^T A' x = 1^T b' = \delta
\]

(Note that there was a missing transpose on the assignment sheet; it should be \( c^T = 1^T A' \) in the definition of \( c \).)

and consequently \( c^T x' < \delta \). We have \( x' \in P \setminus F \). Let us denote the inequalities of \( A'x \leq b' \) as \( a_j^T x \leq b_j \) for \( j = 1 \ldots k \), and rearrange the inequalities such that \( a_j^T x' < b_j' \).

\[
c^T x' = 1^T A' x' = \sum_{j=1}^{k} a_j^T x' = \sum_{j=2}^{k} a_j^T x' < b_j' + \sum_{j=1}^{k} b_j' = 1^T b = \delta
\]

and so \( c^T x' < \delta \).

One has \( F = \{ x \in P \colon A'x = b' \} \). Let us show \( F \subseteq \{ x \in P \colon A'x = b' \} \). Let \( x \in F \). Then by definition of \( F \), we have \( x \in P \), and by definition of \( P \), we have \( A'x \leq b' \). We only need to check \( A'x = b' \). Assume that this is not true, then without loss of generality we have \( a_j^T x < b_j' \). But then the same computation as above shows that \( c^T x < \delta \), which is a contradiction.

Now let us show \( \{ x \in P \colon A'x = b' \} \subseteq F \). Let \( x \in P \), \( A'x = b' \). Then

\[
c^T x = 1^T A' x = 1^T b' = \delta
\]

and therefore \( x \in F \).

Exercise 4 (⋆)

Show the following: A face \( F \) of \( P = \{ x \in \mathbb{R}^n \colon Ax \leq b \} \) is inclusion-wise minimal if and only if it is of the form \( F = \{ x \in \mathbb{R}^n \mid A'x = b' \} \) for some subsystem \( A'x \leq b' \) of \( Ax \leq b \).

Let \( F \) be an inclusion-wise minimal face. Write

\[
F = \{ x \in P \colon A'x = b' \}
\]
where $A' x \leq b'$ is the maximal possible subsystem of $Ax \leq b$ with that property, and let 

$$G = \{x \in \mathbb{R}^n: A' x = b'\}$$

Assume $F \neq G$, then there is a point $z \in G \setminus F$. In particular, $z \notin P$. Furthermore, there exists a point $y \in F$. Consider the line segment parameterized by:

$$w(t) = (1 - t)y + tz, \quad t \in [0, 1]$$

Let $a^T x \leq \beta$ be the first inequality of $Ax \leq b$ that is violated as $w(t)$ moves from $y$ to $z$, and let $t \in [0, 1)$ such that $a^T w(t) = \beta$. Then

$$F' = \{x \in P: A' x = b', a^T x = \beta\}$$

is a face of $P$ by Exercise 2, it is clearly contained in $F$, and it is non-empty because $w(t) \in F'$. Finally, note that $a^T x = \beta$ cannot be contained in the system $A' x = b'$, because $a^T w(t) = \beta$ does not hold for all $t \in [0, 1]$. Therefore, $F'$ is defined by a subsystem of equations that is strictly bigger than any subsystem that defines $F$ (remember that we chose $A' x = b'$ to be maximal!) and so $F' \neq F$. In conclusion, $F'$ is a proper sub-face of $F$, which contradicts the inclusion-wise minimality of $F$. So the assumption was wrong, in fact we have $F = G$.

Let $F$ be a face of $P$ such that $F = \{x \in \mathbb{R}^n: A' x = b'\}$ for a subsystem $A' x \leq b'$ of $Ax \leq b$. Assume that $F$ is not inclusion-wise minimal, i.e. there is a proper sub-face $F' \subsetneq F$. We can write

$$F' = \{x \in P: A' x = b', A'' x = b''\}$$

for a subsystem $A'' x \leq b''$ of $Ax \leq b$. Let $y \in F'$ and $z \in F \setminus F'$. Then the line through $y$ and $z$ is contained entirely in $F$, however there will be one inequality $a^T x \leq \beta$ of the system $A'' x \leq b''$ that is not parallel to the line through $y$ and $z$. This means that the line cannot be entirely contained in $P$. This is a contradiction, and so $F$ must be inclusion-wise minimal.