For an independence system \((S, \mathcal{I})\), define

0. if \(I, J \in \mathcal{I}\) with \(|J| > |I|\), then there exists \(x \in J \setminus I\) such that \(I \cup \{x\} \in \mathcal{I}\).

**Exercise 1**

For the first one, we set \(S = E(G)\) and \(\mathcal{I} = \{I \subseteq S : I\text{ is a subset of an } s - t \text{ path}\}\). For the other two we proceed similarly. It is easy to provide examples where some of the equivalent matroid conditions 0-3 are not satisfied, hence none of them is a matroid.

**Exercise 2**

Assuming \((S, \mathcal{I})\) is an independence system, it is enough to show that 0 holds true if and only if any of 1−3 holds true.

0 \(\iff\) 1 Clear.

0 \(\iff\) 2 One direction is clear. For the other, assume 2 holds true. We prove the statement by induction on \(|I \setminus J|\). When \(|I \setminus J| = 0\), then \(I \subseteq J\) and the statement holds by definition of independence system. So suppose \(|I \setminus J| \geq 1\). Pick \(i \in I \setminus J\). We apply induction on \(I \setminus \{i\}\) and \(J\), as to obtain \(j \in J \setminus I\) such that \(I \cup \{j\} \in \mathcal{I}\). We apply induction again on \(I \setminus \{i\} \cup \{j\}\) and \(J\), as to obtain an element \(j' \in J \setminus I\) with \(I \setminus \{i\} \cup \{j, j'\} \in \mathcal{I}\). We now apply 2 to \(I\) and \(I \setminus \{i\} \cup \{j, j'\}\), and deduce that either \(I \cup \{j\}\) or \(I \cup \{j'\} \in \mathcal{I}\). Since \(j, j' \in J \setminus I\), we deduce the thesis.

0 \(\iff\) 3 One direction is clear. For the other, assume 3 holds true. Then if \(I, J \in \mathcal{I}\) and \(|I| < |J|\), \(I\) cannot be maximal. Suppose by contradiction that \(I' = I \cup \{x\} \in \mathcal{I}\) for some \(x \in (J \cup I)\), but for no \(y \in J \setminus I\). Then condition 3 is violated when we pick \(A = J \cup \{x\}\), since both \(J\) and \(I\) are maximum independent sets within \(A\), but they are of different cardinality by assumption.

Property 3 is often very useful when we want to show that an independence system is a matroid. We will use it frequently in the next exercises.

**Exercise 3**

Clearly \((S, \mathcal{I})\) is an independence system. It is then enough to verify condition 3 from Exercise 2 holds true. Suppose not, i.e. there exist some set \(A\) with two bases \(I, J \subseteq A\) of different cardinality, say \(|I| < |J|\). Clearly \(|I \setminus J|, |J \setminus I| \neq \emptyset\), else the statement is trivial. Let \(M_1\) and \(M_2\) be minimal matchings that cover vertices of \(I\) and \(J\) respectively. Consider the graph \(G' = (V, M_1 \triangle M_2)\). Nodes with degree one are those covered by exactly one of \(M_1\) and \(M_2\), while nodes with degree 2 (resp. 0) are those covered by both matchings with different edges (resp. the same edge). Hence \(G'\) is the disjoint union of even cycles, paths where edges alternate between \(M_1\) and \(M_2\), and isolated vertices. Since \(|J \setminus I| > |I \setminus J|\), there exist some node
of \( J \), say \( v \), that is not matched to any node of \( I \), and that is the extreme point of a path of \( G' \) that ends in a vertex \( u \notin I \). This path \( P \) has one edge from \( M_2 \) more than it has edges from \( M_1 \). Then the matching \( M'_1 = M_1 \Delta M_2 \) covers nodes \( I \cup \{v\} \subset A \), contradicting the fact that \( I \) is a basis.

**Exercise 4**

Clearly \( M \) is an independence system. We show that condition 3 from Exercise 2 holds. Take \( A \subseteq E \), and any basis \( I \) of \( A \) (wrt \( M \)). Then \( I \cap S_1 \in \mathcal{I} \). It is in fact a basis of \( A \) (wrt \( M_1 \)), otherwise there exists \( x \in (S \cap A) \setminus I \) such that \( I \cup \{x\} \subseteq S \) is independent in \( M \), a contradiction. So \( r_1(A \cap S_1) = |I \cap S_1| \) and similarly \( r_2(A \cap S_2) = |I \cap S_2| \). Hence \( |I| = r_1(A \cap S_1) + r_2(A \cap S_2) \), hence it does not depend on \( J \). Hence \( r(A) = r_1(A \cap S_1) + r_2(A \cap S_2) \) is independent of the basis of \( A \) we chose, and we are done. Note that this proof technique lets us, at the same time, prove that an independence system is a matroid and describe its rank function. This will also be useful for next exercises.

**Exercise 5**

The uniform matroid of rank \( t \) over \( n \) elements \( M = (E, \mathcal{I}) \) is such that \( E = \{1, \ldots, n\} \) and \( \mathcal{I} = I \subseteq E \) such that \( |I| \leq t \). It is easy to see that it is a matroid. The partition matroid can be seen as the union of the rank 1 uniform matroids over \( E_i \). Using the result from the previous exercise, we conclude it is a matroid.

**Exercise 6**

If an independence system is not a matroid, there exists \( A \subseteq E \) that violates property 3 from Exercise 2. In particular, let \( I \) be a maximal independent set contained in \( A \) not of maximum cardinality, while let \( J \) be a maximal independent set of \( A \) of maximum cardinality. Let \( c_e = 1 + \varepsilon \) for \( e \in I \), \( c_e = 1 \) for \( e \in A \setminus I \), and \( c_e = 0 \) otherwise, where \( \varepsilon > 0 \) to be fixed. Then the greedy algorithm will output \( I \). For \( \varepsilon \) small enough (e.g., \( \varepsilon < 1/|I| \)), we have

\[
c(J) = |J| \geq |I| + 1 > |I|(1 + \varepsilon) = c(I),
\]

hence \( I \) is not an independence system of maximum weight.

**Exercise 7**

Take a smallest independent set \( J \) such that \( J \cup \{x\} \) contains two circuits \( C_1 \) and \( C_2 \) for some \( x \notin J \). By the minimality of \( J \) we deduce that \( J \cup \{x\} = C_1 \cup C_2 \). Moreover, there exist \( a, b \) with \( a \in C_1 \setminus C_2 \) and \( b \in C_2 \setminus C_1 \) (else one is contained in the other, contradicting the fact that they are circuit). Let \( J' = C_1 \cup C_2 \) and note that \( J' \setminus \{a, b\} \) is an independence set, since otherwise it would contain a circuit \( C \), and \( J \setminus \{a\} \cup \{x\} \) would contain the circuits \( C_2 \) and \( C \), contradicting the minimality of \( J \). Hence \( J' \setminus \{a, b\} \) and \( J \) are maximal independent sets contained in \( J \cup \{x\} \), but they have different cardinalities, contradicting property 3 from Exercise 2.

**Exercise 8**

\( M^* \) is usually called dual matroid. It is easy to check that it is indeed an independence
system. We now show it satisfies property 3 from Exercise 2, with an approach similar to the one used in Exercise 4, i.e. picking a maximal independent set $J^*$ (wrt $M^*$) of some $A \subseteq S$ and showing that $|J^*|$ is independent of the specific $J^*$ we picked. This will also characterize the rank function of $M^*$. Hence, let $A$ and $J^*$ be as above. This means that $r(S \setminus J^*) = r(S)$. Consider a basis $B$ (wrt $M$) of $S \setminus A$ and extend it to a basis $B'$ (again wrt $M$) of $S \setminus J^*$ (this can be done since $M$ is a matroid, hence satisfies property 0). We claim that $A \setminus J^* \subseteq B'$. Suppose not, then $x \not\in B'$ for some $x \in A \setminus J^*$. Then
\[ r(S) = r(S \setminus J^*) = r(B') = r(S \setminus (J^* \cup \{x\})) , \]
contradicting the fact that $J^*$ is an independent set of maximum size (wrt $M^*$) contained in $A$. Hence,
\[ r(S) = r(S \setminus J^*) = |B'| = |B| + |A \setminus J^*| = r(S \setminus A) + |A| - |J^*| , \]
from which we deduce that $|J^*| = r(S \setminus A) - r(S) + |A|$, hence does not depend on $J^*$, as required.