UNIVERSITÄT PADERBORN
Die Universität der Informationsgesellschaft
Fakultät für Elektrotechnik, Informatik und Mathematik

Diplomarbeit

Combinatorial abstractions for the diameter of polytopes

Paderborn, den 2. Oktober 2008

vorgelegt bei
Prof. Dr. Friedrich Eisenbrand

Nicolai Hähnle
Fechtelerstr. 17
33100 Paderborn

Matrikelnummer: 6285390
E-Mail: prefect@upb.de

Contents

1 Introduction

Beginning in the 1940s, linear programming has been developed as a powerful tool for modelling and solving real world optimization problems [Dan63][Chv83]. Quoting [MG07], a linear program is the problem of maximizing a given linear function over the set of all vectors that satisfy a given system of linear equations and inequalities. Such vectors are called feasible solutions. Every linear program can be transformed to the form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

where $c \in \mathbb{R}^d$ defines the linear objective function and the matrix $A \in \mathbb{R}^{n \times d}$ and the vector $b \in \mathbb{R}^n$ describe the set of feasible solutions in $\mathbb{R}^d$. The inequality $Ax \leq b$ is a shorthand for the system of linear inequalities $a_i x \leq b_i$, $i = 1 \ldots n$, where $a_1, \ldots, a_n$ are the rows of A. It follows that the set of feasible solutions is a polyhedron P in d-dimensional space. If none of the inequalities $a_i x \leq b_i$ are redundant and P is full dimensional, the polyhedron P has n facets. If P has vertices, at least one of its vertices is an optimal solution to the linear program.

The simplex method, developed by George Dantzig, was the first algorithm for solving linear programs. It works by starting at an arbitrary vertex of P and then successively moving to a neighbouring vertex that is better with respect to the objective function. The simplex method has found an optimal solution when it can no longer find a better neighbouring vertex. Thus one can think of the simplex method as performing a walk on the graph of the polyhedron. This is a great simplification of how the simplex method works in practice, but it is sufficient for our purposes. The obvious question one should ask is whether the simplex method is efficient.

The common answer in the literature is that the simplex method works well in practice, but the question of whether its worst case running time can be bounded by a polynomial in n and d is still open. The underlying problem is that the number of vertices of the polyhedron described by $Ax \leq b$ cannot be bounded by a polynomial in n and d, so one cannot efficiently obtain an explicit description of the polyhedral graph. In fact, the number of vertices can be on the order of $n^{\lfloor d/2 \rfloor}$, for example when the polyhedron is the polar of a cyclic polytopes [Gal63]. Of course, the worst case running time of the simplex method could still be small despite this fact. Furthermore, it would be interesting to know why the simplex method works so well in practice despite its bad worst case behaviour. These questions have lead to some interesting developments. For example, it has been shown that the expected running time of certain variants of the simplex method is small when the polyhedron given by A and b is taken from some particular random distributions [AKS87][Bor87]. Even better, the expected running time of a variant of the simplex method is polynomial in the worst case if a small random perturbation of the input is allowed [ST04][Ver06].

We also want to understand the theoretical limits of the simplex method. Even if the simplex method were somehow able to always pick the best possible neighbouring vertex, its efficiency is limited by the structure of the graph of the polyhedron. In particular, the diameter of this graph yields an immediate lower bound for the running time of the simplex method.[1] Therefore, we are interested in the maximum diameter $\Delta_U(d,n)$ of a d-dimensional polyhedron with n facets.

In 1957, Warren M. Hirsch conjectured that $\Delta_U(d,n)$ is bounded from above by $n-d$ [Dan63, p. 160]. This so called *Hirsch conjecture* was shown to be false for unbounded polyhedra. The lower bound of $\Delta_U(d,n) \geq n-d+\min\{\lfloor \frac{d}{4} \rfloor, \lfloor \frac{n-d}{4} \rfloor\}$ given in [KW67] is larger than $n-d$, though still linear. The Hirsch conjecture is true for all bounded polyhedra of dimension $d \leq 5$ [KW67].

A number of related conjectures have been published, corresponding to the various ways in which researchers have attempted to attack the Hirsch conjecture. The d-step conjecture, also due to Hirsch, states that $\Delta_U(d,2d) \leq d$. The nonrevisiting path conjecture by Klee and Wolfe [Kle66][Hol03] states that any two vertices of a polyhedron can be connected by a path that never reenters a facet it has previously left. These conjectures are all equivalent in some sense [KK87]. Unfortunately, despite the great variety of techniques that have been used, settling the bounded Hirsch conjecture seems as elusive as ever.

Given the apparent difficulty of the problem, it is natural to attempt to prove weaker statements. On the one hand, researchers have investigated special classes of polyhedra. For example, the Hirsch conjecture is true for 0/1-polytopes [Nad89] and certain transportation polytopes [BR93]. Kalai has found a polynomial upper bound on the diameter of polytopes with a maximal number of vertices such as the polar polytopes of cyclic polytopes [Kal91]. For this thesis, however, I have restricted myself to the more general theory. Larman proved that $\Delta_U(d,n) \leq 2^{d-3}n$ for $d \geq 2$ [Lar70], improving an earlier result due to Klee which was published in [Grü67, chapter 16]. Kalai found the first subexponential upper bound [Kal92] and subsequently improved it to $\Delta_U(d,n) \leq n^{\log d+2}$ with a surprisingly simple proof in [KK92].

The interesting thing about these general upper bounds is that they do not use the full geometry of the problem. In fact, as I will show in this thesis, these upper bounds hold in much more general settings. Analyzing the diameter of polyhedra in abstract settings is not a new idea. Adler and Dantzig have done this very explicitly in [AD74] and Kalai also mentions a combinatorial abstraction in [Kal92]. I was able to show that the general upper bounds cited above hold in an abstract setting in which a superlinear lower bound on the diameter exists.

There are bounds on the diameter of polyhedra that cannot be translated into this abstract setting. For example, the bound in the case where $d \leq 5$ can be proven in Adler's abstraction but fails in the new abstractions I present in this thesis. Furthermore, results

[1] This is true as long as the choice of initial vertex does not depend on the objective function.

for specific classes of polyhedra do not necessarily translate into such abstract frameworks. It is unclear whether Kalai's polynomial upper bound on the diameter of polyhedra with a maximum number of vertices [Kal91] can be translated, for example. Thus it becomes interesting to figure out which additional aspects of the geometry of polyhedra these results use, but this is beyond the scope of my thesis.

Finally, it is important to note that the algorithmic side of linear programming has also been investigated from such an abstract perspective. Sharir and Welzl [SW92] described an algorithm for linear programming that can be understood in an abstract framework. Matousek [Mat94] proved that their analysis was essentially tight within the abstract framework, while Gärtner proved in [Gä98] that the analysis can be improved if the geometry of linear programs is used in addition to the properties of the abstract framework. It is an interesting question whether one can find a useful relationship between this abstract framework and the abstractions I present in this thesis.

1.1 About this thesis

As often seems to be the case in mathematics, this thesis is largely written backwards. After all, I do not want to bore you with the details of how I ended up with my results, fascinating though those details may be for myself. Instead, I simply want to present the results to you. Thus, after an introductory chapter on notation which you should feel free to skip if you are familiar with the subject,[2] I will immediately present the definitions of the combinatorial abstractions that I feel are relevant and interesting in chapter 3. Of course, I will also give you an idea of why I feel that these abstractions are relevant and interesting and I will establish some basic relationships between them. You will also find the proofs for upper bounds on the diameter in this chapter.

Then, in chapter 4, I will make a detour to combinatorial designs that might at first seem completely unmotivated, but remember that large parts of this thesis are written backwards. In fact, that chapter will introduce a new problem of combinatorial design, the problem of finding families of disjoint coverings, which will play a central role in chapter 5. In that chapter, I will tell you how to construct objects of superlinear diameter in certain combinatorial abstractions, which reinforces the point that those abstractions really are interesting. Being so interesting, they tempted me to investigate a number of special cases, my results concerning one of which are presented in chapter 6.

Once you have read everything before it, chapter 7 will hopefully give you a nice feeling of closure despite containing little additional information. Of course, you might choose to read only chapter 7 and ignore the rest, but besides being less interesting that would only reinforce certain demotivating stereotypes, so please don't.

[2] But do not hesitate to go back to the sections on multisets and set families if you get lost in the details of later chapters. The proofs are quite dense at times (you should have seen the early versions!), and those sections in chapter 2 are supposed to help.

1.2 Acknowledgments

I want to thank Prof. Friedrich Eisenbrand for giving me the chance to write this thesis on the problems that interest me. I also want to thank Thomas Rothvoß for enlightening discussions about the world of abstract and convex polyhedra. Finally, as a representative for the very long list of people who have brought me in so many small – and sometimes huge – ways to where I am today, I would like to thank my former flatmate Mohammed who is almost, but not quite, entirely capable of making people choke from laughter.

2 Notation and basic facts

In this chapter, I will give a quick introduction to the basic objects which are used throughout this thesis. I assume that the reader is familiar with basic mathematical notation.

Furthermore, I assume that the reader is familiar with basic notions of linear algebra, geometry (such as linear maps, affine maps, closed and open sets), and modular arithmetic. As a shorthand, I denote by $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$ the ring of integers modulo n. For the purpose of this text, the natural numbers $\mathbb{N}$ include the number zero, that is

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Given a set F, the power set of F is written as 2^F. For a natural number d, $\binom{F}{d}$ is the set of d-element subsets of F.

2.1 Multisets

Informally, a *multiset* is a collection of objects where – unlike in a set – each object may appear more than once. I only consider finite multisets, that is multisets that contain a finite number of objects and each object appears only a finite number of times in a multiset.

Sometimes, it is instructive to think of a multiset as a vector of natural numbers where each component of the vector indicates the number of appearances of one object. With this in mind, the following definition and notation makes sense.

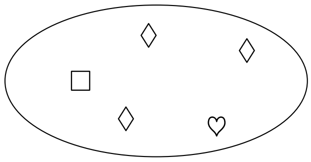
Definition 2.1.1. *Let F be a finite, non-empty set. The set $\mathbb{N}^F$ is the* set of finite multisets over the groundset F. *Formally, the set $\mathbb{N}^F$ is the appropriate set of tuples containing natural numbers or the set of functions $f : F \to \mathbb{N}$.*

Let $v \in \mathbb{N}^F$. Then for all $a \in F$ we denote by v_a the a-th component of the vector v or, equivalently, the multiplicity of a in v, i.e. the number of times that the object a appears in the multiset v.

I will use curly braces for enumerative definitions of multisets, e.g. $\{\square, \triangle, \square\}$ denotes a multiset containing two squares and one triangle. There is potential for confusion with normal sets here, so I have taken care to ensure that the meaning of such a definition is clear from the context. Sometimes, I have taken the liberty of leaving out curly braces in examples so as to make them easier to read. As an example, $\square\triangle\square$ means the same thing as $\{\square, \square, \triangle\}$ in an appropriate context. The order of elements is irrelevant.

Since multisets will usually play the role of vertices in this text, I have chosen to use lowercase letters u, v, and so on for variables that contain multisets. Elements of multisets will usually be denoted with lowercase letters a, b, and so on.

Example 2.1.2. Let $F = \{\square, \triangle, \lozenge, \heartsuit\}$ be a groundset. The multiset $v := \{\square, \lozenge, \heartsuit, \lozenge, \lozenge\}$ over the groundset F can be understood as a collection of objects and as a vector containing multiplicities.



$\square \times 1$
$\triangle \times 0$
$\lozenge \times 3$
$\heartsuit \times 1$

$$v = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix} \in \mathbb{N}^F$$

When we understand v as a vector in this sense, we can refer to its components. For example, we have $v_\lozenge = 3$ and $v_\triangle = 0$.

Of course, one can perform the usual set theoretic operations with multisets, though some care must be taken to ensure the operations do the right thing in the face of multiplicities.

Definition 2.1.3. *Let u and v be multisets over a common groundset F.*

a) *The* cardinality $|u|$ *is the number of objects in u. Formally,* $|u| = \sum_{a \in F} u_a$.

b) *We say that u is a subset of v, denoted $u \subseteq v$, if $u_a \le v_a$ for all $a \in F$.*

c) *The* union $u \cup v$ *is defined by* $(u \cup v)_a = u_a + v_a$ *for all $a \in F$. One can say that the union of multisets is always a* disjoint union.

d) *The* intersection $u \cap v$ *is defined by* $(u \cap v)_a = \min\{u_a, v_a\}$ *for all $a \in F$.*

e) *Suppose $u \subseteq v$. Then the* set difference $v \setminus u$ *is defined by* $(v \setminus u)_a = v_a - u_a$ *for all $a \in F$. The expression $v \setminus u$ is undefined if $u \not\subseteq v$.*

f) *The set* $\left(\!\left({F \atop k} \right)\!\right) := \{v \in \mathbb{N}^F \mid |v| = k\}$ *is the set of k-element multisets over F.*

The definitions given here are for multisets over a common groundset, but this is not a limitation in any way. After all, given groundsets $F \subset G$ any multiset v over F is also a multiset over G in a straightforward way.

When I use multisets as index sets in summations, multiple appearances of objects should be taken into account. Formally, given a multiset v over a groundset F and an expression $f(a)$ I set $\sum_{a \in v} f(a) := \sum_{a \in F} v_a f(a)$. This will be used in chapter 6 where I consider multisets containing natural numbers.

Example 2.1.4. Let $F = \{0,1,2,3\} \subset \mathbb{N}$ and let $v = \{2,2,3\} \in \mathbb{N}^F$ be a multiset. Then $\sum_{a \in v} a = 2 + 2 + 3 = 7$.

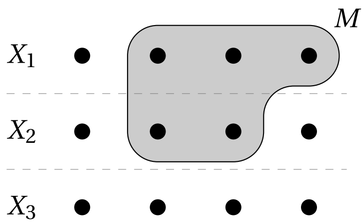
2.2 Families of sets

In the constructions of chapters 4 and 5 we will often have to deal with subsets of a groundset X that arises from the disjoint union of smaller groundsets X_j.

Definition 2.2.1. *Let $d \geq 1$. Let $(X_1, \ldots, X_d)$ be a tuple of disjoint sets and set $X := X_1 \cup \ldots \cup X_d$. Furthermore, let $M \subseteq X$. Then M is called a $(v_1, \ldots, v_d)$-set with respect to $(X_1, \ldots, X_d)$ if $|M \cap X_j| = v_j$ for all $1 \leq j \leq d$. In short and when the X_j are clear from the context, M is simply called a v-set.*

You can interpret v both as a vector with d components over the natural numbers and as a multiset over the groundset $\{1, \ldots, d\}$. This is not an accident.

Example 2.2.2. Suppose we have a tuple (X_1, X_2, X_3) of disjoint sets with $|X_j| = 4$ for all j. Furthermore, let $v := (3, 2, 0)^T$. The picture visualizes the elements of $X = X_1 \cup X_2 \cup X_3$ as points. One possible v-set M is shown as well.



Note how v can also be interpreted as the multiset $\{1, 1, 1, 2, 2\}$. If you project M onto a vertical line, you can intuitively identify this projection with the multiset v.

Setting $F := \{1, \ldots, d\}$, his last observation leads to the definition of the *projection function* $\pi : 2^X \to \mathbb{N}^F$ where $\pi(M) = v$ if and only if M is a v-set with respect to $(X_1, \ldots, X_d)$. One can easily verify that π is well-defined.

Definition 2.2.3. *Let X and Y be disjoint groundsets and let $\mathscr{F} \subseteq 2^X$ and $\mathscr{G} \subseteq 2^Y$ be collections of subsets of X and Y, respectively. Then the unordered product $\mathscr{F} \otimes \mathscr{G}$ is defined by $\mathscr{F} \otimes \mathscr{G} := \{u \cup v \mid u \in \mathscr{F}, v \in \mathscr{G}\}$.*

Example 2.2.4. Let $X = \{1, 2, 3, 4\}$ and $Y = \{\square, \triangle, \lozenge\}$. Furthermore, let

$$\mathscr{F} = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$$
$$\mathscr{G} = \{\{\square\}, \{\lozenge\}\}$$

Then the unordered product is obtained by combining every element of $\mathscr{F}$ with every element of $\mathscr{G}$.

$$\mathscr{F} \otimes \mathscr{G} = \{\{1, 2, \square\}, \{1, 2, \lozenge\}, \{1, 3, \square\}, \{1, 3, \lozenge\}, \{2, 4, \square\}, \{2, 4, \lozenge\}\}$$

Since reading all those curly braces can become a bit awkward I will often omit the innermost curly braces in future examples. I would then write the line above as:

$$\mathscr{F} \otimes \mathscr{G} = \{12\square, 12\lozenge, 13\square, 13\lozenge, 24\square, 24\lozenge\}$$

The definition of unordered products extends quite naturally to unordered products of arbitrarily many collections of subsets. So given disjoint groundsets $X_1, \ldots, X_d$ and collections $\mathscr{F}_1, \ldots, \mathscr{F}_d$ with $\mathscr{F}_j \subseteq 2^{X_j}$ for all j, we have

$$\mathscr{F} := \bigotimes_{j=1}^{d} \mathscr{F}_j = \left\{ \bigcup_{j=1}^{d} u_j \mid u_j \in \mathscr{F}_j \right\}$$

Suppose the subsets contained in each $\mathscr{F}_j$ are of uniform cardinality, that is for all j there exists a number v_j such that $|u| = v_j$ for all $u \in \mathscr{F}_j$. Then every subset $u \in \mathscr{F}$ is a v-set with respect to $(X_1, \ldots, X_d)$. I will use this "recipe" extensively to construct collections of v-sets.

2.3 Graphs and matchings

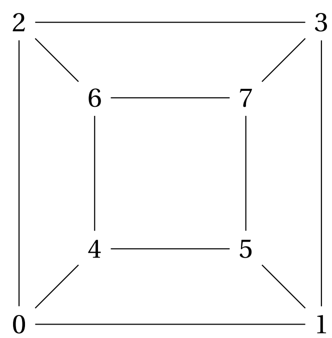
Definition 2.3.1. *An (undirected)* graph *is a pair* $G = (V, E)$*, where* V *is a nonempty, finite set of* vertices *and* $E \subseteq \binom{V}{2}$ *is a set of* edges. *Two vertices* $u, v \in V$ *are called* adjacent *if they are connected by an edge in the graph, that is* $\{u, v\} \in E$.

A path *in* G *is a sequence* $P = (v_0, v_1, \ldots, v_k)$ *of vertices such that* v_j *and* v_{j+1} *are adjacent for all* $0 \leq j < k$. *The number* k *is called the* length of P.

The distance $d(u, v)$ *between two vertices* $u, v \in V$ *is the length of a shortest path connecting* u *and* v*, or* ∞ *if no such path exists. The* diameter $\operatorname{diam}(G)$ *of a graph is the maximum distance between any pair of vertices, i.e.* $\operatorname{diam}(G) := \max_{u,v \in V} d(u, v)$.

Example 2.3.2. The following is a formal and a visual representation of a graph $G = (V, E)$ which happens to be the graph of a 3-dimensional cube.

$$\begin{aligned} V &:= \{0, 1, 2, 3, 4, 5, 6, 7\} \\ E &:= \{\{0,1\}, \{0,2\}, \{0,4\}, \{1,3\}, \{1,5\}, \{2,3\}, \\ &\quad \{2,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{5,7\}, \{6,7\}\} \end{aligned}$$



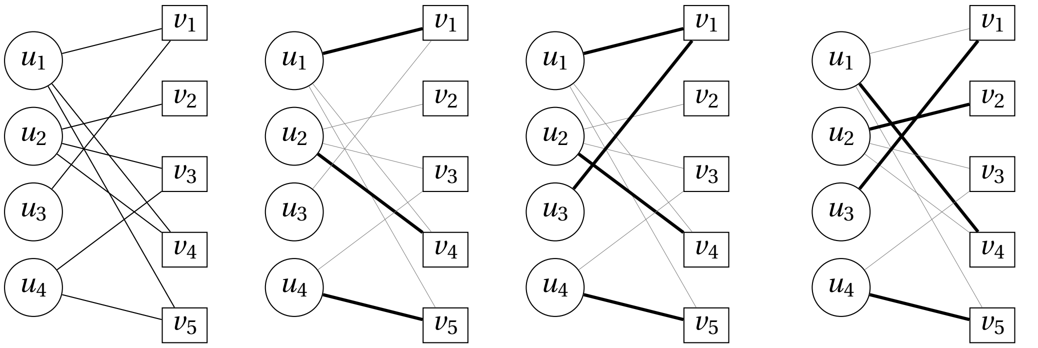
The diameter of G is 3.

In chapter 4, we will use bipartite graphs for some constructions.

Definition 2.3.3. *A graph* $G = (U \cup V, E)$ *is called* bipartite *if* U *and* V *are disjoint and all edges of* G *connect a vertex in* U *with a vertex in* V*, that is all* $e \in E$ *are of the form* $e = \{u, v\}$ *where* $u \in U$ *and* $v \in V$*.*

A subset $M \subseteq E$ *of edges is called a* matching (in G) *if all edges in* M *are disjoint, that is every vertex has at most one incident edge in* M*. A matching is called* U-perfect *if for every vertex* $u \in U$ *there is an edge in* M *incident to* U*.*

Example 2.3.4. The following pictures show, from left to right, a bipartite graph $G = (U \cup V, E)$, the graph G and a subset of edges that is a matching in G, the graph G and a subset of edges that is *not* a matching, and the graph G and a subset of edges that is a U-perfect matching.



graph G matching not a matching U-perfect matching

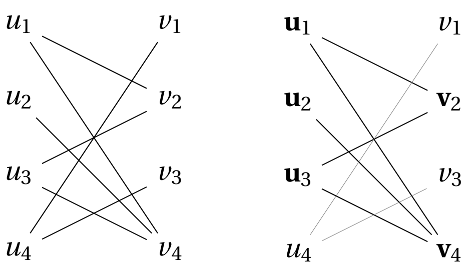
Let $A \subseteq U$ be a set of vertices. The *neighbourhood of* A, denoted by $N(A)$, is the set of vertices that are neighbours of vertices in A. In other words, $N(A)$ contains those vertices that are connected to a vertex in A by an edge of the graph.

The following theorem due to Hall [Hal35], which was originally a statement about set systems and representatives, gives a useful criterion for the existence of U-perfect matchings which we will use in chapter 4 for some lower bound constructions.

Theorem 2.3.5 (Marriage Theorem)**.** *Let* $G = (U \cup V, E)$ *be a bipartite graph. Then there exists a* U*-perfect matching in* G *if and only if for every* $A \subseteq U$*,* $|N(A)| \geq |A|$*.*

Proof. See [Die00, theorem 2.1.2] for three (!) very different proofs. □

Example 2.3.6. Consider the bipartite graph in the following picture.



graph G A and $N(A)$

The neighbourhood of $A = \{u_1, u_2, u_3\}$ contains only the vertices v_2 and v_4. Therefore, there is no U-perfect matching in this graph.

2.4 Polyhedra and polytopes

In this section, I will list the usual definitions and some basic results about polyhedra that are relevant for my work. My notation and terminology follows [Zie95]. I would also like to refer the reader there for proofs of the cited results.

Definition 2.4.1. *A set $P \subset \mathbb{R}^{d'}$ is called a* polyhedron *if it is the intersection of finitely many halfspaces. It is called a* polytope *if it is bounded. The* dimension *of P is the dimension of the smallest affine space that contains it.*

Note that the dimension of a polyhedron $P \subset \mathbb{R}^{d'}$ may be smaller than d'. For example, a 2-dimensional polyhedron (a polygon) can live in a 3-dimensional space. However, I will usually assume that polyhedra are full-dimensional. This is without loss of generality because you can simply restrict your attention to the appropriate affine subspace.

Definition 2.4.2. *Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional polyhedron. Let $a \in \mathbb{R}^{1\times d'}$ and $b \in \mathbb{R}$ such that P lies on one side of the hyperplane defined by $ax = b$, say $ax \leq b$ for all $x \in P$. Then the intersection*

$$P \cap \{x \in \mathbb{R}^{d'} | ax = b\}$$

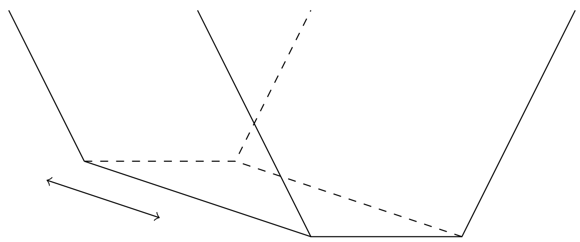
is called a face *of P.*

0*-dimensional faces are called* vertices*,* 1*-dimensional faces are called* edges*, and* $(d-1)$*-dimensional faces are called* facets *of P.*

Note that the empty set and P itself are also faces of P, because the definition allows $a = 0$. Furthermore, faces of P are polyhedra.

Definition 2.4.3. *A polyhedron is called* pointed *if it has a vertex.*

All polytopes are pointed. However, polyhedra in general need not be pointed. To see this, simply extrude an unbounded polyhedron in an additional dimension as in the picture below – just imagine that the polyhedron stretches out into infinity in both directions indicated by the arrow.



The smallest proper faces of this polyhedron are lines. I will only concern myself with pointed polyhedra, because the following definitions do not make sense otherwise.

Definition 2.4.4. *Let $P \subset \mathbb{R}^{d'}$ be a pointed polyhedron. The* graph *of P is the graph $G(P) := (V(P), E(P))$, where:*

$$V(P) := \{x \mid \{x\} \text{ is a vertex or } 0\text{-face of } P\}$$
$$E(P) := \{\{x, y\} \mid \text{the line segment } [x, y] \text{ is an edge or } 1\text{-face of } P\}$$

The diameter $\operatorname{diam}(P)$ of P *is the diameter of $G(P)$. Given a fixed dimension d and number of facets n, we define $\Delta_B(d, n)$ to be the maximum diameter of a d-dimensional polytope with n facets. Similarly, we define $\Delta_U(d, n)$ to be the maximum diameter of a d-dimensional polyhedron with n facets.*[1]

In more general theory, $G(P)$ is also called the 1-*skeleton* of P as it describes the combinatorial structure of all 1-dimensional and all lower-dimensional faces of P. If F is a face of P, then $G(F)$ is a subgraph of $G(P)$ in a natural way. $G(P)$ is always connected and $\Delta_U(d, n)$ and $\Delta_B(d, n)$ are finite.

[1] The B stands for "bounded" while the U stands for "unbounded".

3 Combinatorial abstractions for the diameter problem

In this chapter, I will review combinatorial abstractions that have been used in the past to study the diameter problem. I will introduce abstract polyhedral graphs and blueprints and I will show how they are related to the previously known abstractions. Finally, I will quickly review the upper bounds for general dimension and how they can be proved in the setting of abstract polyhedral graphs and blueprints.

3.1 Ultraconnected families of d-sets and abstract polytopes

Adler and Dantzig were the first to study the diameter of polytopes using a combinatorial abstraction, which they called abstract polytope [AD74]. Kalai mentions an abstraction which he calls ultraconnected families of d-sets in [Kal92]. He states that his upper bounds have essentially been found in this abstract context which turns out to be a relaxation of abstract polytopes.

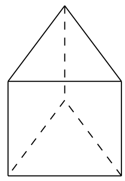
Definition 3.1.1. *Let $1 \leq d \leq n$ be integers. Let $\mathscr{F}$ be a set of n symbols and let $\mathcal{V}$ be a family of d-subsets of $\mathscr{F}$. We call the elements of $\mathcal{V}$* vertices. *We say that $u, v \in \mathcal{V}$ are* adjacent *if $|u \cap v| = d - 1$. This defines a graph structure on $\mathcal{V}$ which is implied when we use terminology from graph theory in relation to $\mathcal{V}$. For example,* $\operatorname{diam}(\mathcal{V})$ *is the diameter of the implied graph. We say that $\mathcal{V}$ is* ultraconnected *if for every pair $u, v \in \mathcal{V}$ there is a path from u to v such that $(u \cap v) \subseteq w$ for every vertex w on the path. We also call d the* dimension *of $\mathcal{V}$.*

Intuitively, the symbols in the set $\mathscr{F}$ correspond to the facets of a convex polyhedron. The vertices $\mathcal{V}$ correspond to the vertices of a (pointed) convex polyhedron. More generally, a subset of symbols corresponds to the intersection of the corresponding facets of a polyhedron or, equivalently, to a face of the polyhedron. This should also explain the motivation for the definition of adjacency: Two vertices are adjacent if they lie in a common edge, and an edge is the intersection of $d - 1$ facets. This intuition will be justified further by actual proofs in section 3.4. For the time being, I will appeal to the reader's ability to generalize the following example.

Remark 3.1.2. I will always assume $1 \leq d \leq n$ throughout the rest of the text.

I decided to deviate from Kalai's terminology where I felt it made sense. For example, Kalai calls the elements of $\mathscr{F}$ vertices, which is inconsistent with the terminology used by Adler for abstract polytopes. Calling the elements of $\mathscr{F}$ "vertices" could also be confusing, as they are in fact related to the facets of polyhedra and not to their vertices.

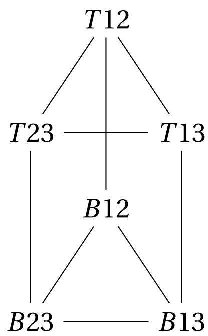
Example 3.1.3. Consider the following 3-dimensional prism.



Suppose the top and bottom facets are labelled T and B, respectively, while the three side facets are labelled from 1 to 3, giving us the set $\mathscr{F} = \{T, B, 1, 2, 3\}$ of symbols. Taking for each vertex the three facets that contain it, we get the following ultraconnected family of 3-sets (as mentioned earlier, I am taking the liberty of writing $T12$ instead of $\{T, 1, 2\}$, and so on):

$$\mathscr{V} = \{T12, T23, T13, B12, B23, B13\}$$

Via the definition of adjacency, these sets implicitly form the vertices of the following graph:



As one can easily see, this is exactly the graph of the original prism. Note how the fact that the prism is simple (that is, every vertex lies on exactly three facets) was important in the construction – this will play a role again later, in the proof that the maximum diameter of ultraconnected sets is an upper bound for the maximum diameter of polyhedra of the same "type".

Definition 3.1.4. *An ultraconnected family $\mathscr{V}$ of d-sets is an* abstract polyhedron *if every $(d-1)$-subset is contained in at most two vertices of $\mathscr{V}$. It is called an* abstract polytope *if every $(d-1)$-subset is contained in no vertex or in exactly two vertices of $\mathscr{V}$. I also use the terms (d, n)-abstract polyhedron and (d, n)-abstract polytope to give the dimension and number of symbols explicitly.*

This definition spells out the geometric fact that an edge of a convex polytope contains exactly two vertices, while unbounded edges of a (pointed) convex polyhedron contain only one vertex. One can easily see that the ultraconnected family of example 3.1.3 is an abstract polytope.

I would like to point out that Adler did not define abstract polyhedra. However, they are a very simple and straightforward generalization that shows in a fascinating way how a seemingly innocent change in the definition can have a major impact on what we can prove about it. A crucial lemma on a-avoiding paths [ADM74] does not hold for abstract polyhedra. In fact, the Hirsch conjecture in dimension 4 and 5 is true for abstract polytopes but not for abstract polyhedra [KW67].

Definition 3.1.5. *I denote the maximum diameter of an ultraconnected family of d-subsets with n symbols with $\Delta_{UF}(d,n)$. (Recall that the diameter is the diameter of the implied graph.) Similarly, I denote the maximum diameter of a (d,n)-abstract polyhedron with $\Delta_{AU}(d,n)$ and the maximum diameter of a (d,n)-abstract polytope with $\Delta_{AB}(d,n)$.*[1]

A very simple relationship between these quantities can be observed immediately.

Proposition 3.1.6. $\Delta_{AB}(d,n) \le \Delta_{AU}(d,n) \le \Delta_{UF}(d,n)$ *for all d and n.*

Proof. Clearly, every abstract polytope is also an abstract polyhedron of the same type. So choose a (d,n)-abstract polytope that achieves diameter $\delta = \Delta_{AB}(d,n)$. Since this abstract polytope is also a (d,n)-abstract polyhedron, we have $\Delta_{AU}(d,n) \ge \delta$.

Similarly, every abstract polyhedron is an ultraconnected family by definition, which shows $\Delta_{AU}(d,n) \le \Delta_{UF}(d,n)$. □

3.2 Abstract polyhedral graphs

Both of the abstractions shown so far have an implicit adjacency structure. Thomas Rothvoß suggested a new abstraction that allows the adjacency structure to be given more explicitly. We have refined this abstraction further, which resulted in the following definition.

Definition 3.2.1. *Let $1 \le d \le n$ be integers. Let $\mathscr{F}$ be a set of n symbols and let $\mathcal{V}$ be a family of d-subsets of $\mathscr{F}$. Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a partition of $\mathcal{V}$, that is $\mathcal{V} = \mathscr{L}_1 \cup \ldots \cup \mathscr{L}_\ell$ and the $\mathscr{L}_j$ are mutually disjoint and nonempty. The sets $\mathscr{L}_j$ are called* levels. *We call d the* dimension *of $\mathscr{L}$.*

We say a set $u \subset \mathscr{F}$ of symbols is alive *or* covered *in level $\mathscr{L}_j$ if there is a vertex $v \in \mathscr{L}_j$ such that $u \subseteq v$.*

*Then $\mathscr{L}$ is a (d,n)-*abstract polyhedral graph *(or just* abstract polyhedral graph*) if the following connectedness condition holds: $\forall 1 \le i < j < k \le \ell$ and for all $u \in \mathscr{L}_i, v \in \mathscr{L}_k$ there is a $w \in \mathscr{L}_j$ such that $(u \cap v) \subseteq w$, i.e. $u \cap v$ is alive in $\mathscr{L}_j$.*

The diameter *of $\mathscr{L}$ is defined as* $\operatorname{diam}(\mathscr{L}) := \ell - 1$. *The maximum diameter of a (d,n)-abstract polyhedral graph is denoted by $\Delta_{APG}(d,n)$, that is*

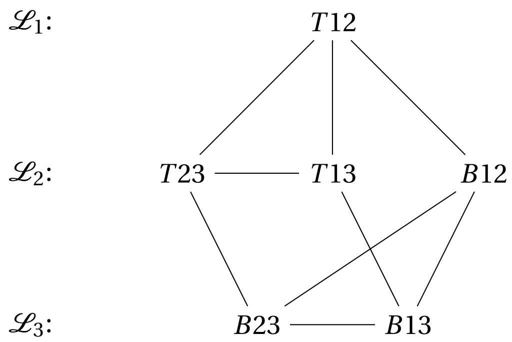
$$\Delta_{APG}(d,n) := \max\{\operatorname{diam}(\mathscr{L}) \mid \mathscr{L} \text{ is a } (d,n)\text{-APG}\}$$

An abstract polyhedral graph is complete *if its vertex set $\mathcal{V}$ is the set $\binom{\mathscr{F}}{d}$ of all d-subsets of $\mathscr{F}$. The maximum diameter of a complete (d,n)-abstract polyhedral graph is denoted by $\overline{\Delta_{APG}}(d,n)$.*

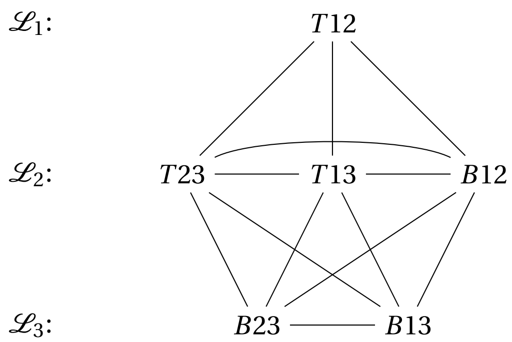
[1] Again, B and U stand for "bounded" and "unbounded", respectively.

Note that even though this definition does not mention adjacency, it does contain the notion of connectedness. In fact, it might help to think about vertices on adjacent levels as being adjacent, as one can then understand abstract polyhedral graphs as a relaxation of ultraconnected sets. The connnectedness condition then says that there has to be a "path" connecting any two vertices without skipping any levels while satisfying the condition for ultraconnectedness.

Example 3.2.2. Recall the abstract polytope of the 3-dimensional prism that was shown in example 3.1.3. In the following drawing, I have rearranged the vertices of the implied graph by looking at their distance from the vertex $T12$. The resulting partition of $\mathcal{V}$ into levels is an abstract polyhedral graph. Lemma 3.4.2 shows that this is not a coincidence. For comparison, you can still see the edges that are implied by the definition of adjacency from ultraconnected families.



Of course, the notion of adjacency that is relevant for the definition of abstract polyhedral graphs is given entirely by the levels. Vertices are adjacent if they are in the same level or in neighbouring levels. The following picture shows this structure of adjacency.



Also note that the diameter of the APG is 2, which is equal to the diameter of the original prism.

Example 3.2.3. The set of vertices $\mathcal{V}$ underlying an abstract polyhedral graph need not be an ultraconnected set. In fact, using the notion of adjacency from definition 3.1.1, the implied graph need not be connected at all, as can be seen in the following trivial (3,7)-APG:

$$\begin{aligned} \mathscr{L}_1 &= \{abc\} \\ \mathscr{L}_2 &= \{ade\} \\ \mathscr{L}_3 &= \{afg\} \end{aligned}$$

Note that, because the intersection of any two vertices contains fewer than two symbols, no two vertices are adjacent in the sense that is relevant for ultraconnected families.

Example 3.2.4. The following is a largest known abstract polyhedral graph of dimension 2 and 8 symbols.

$$\begin{aligned} \mathscr{L}_1 &= \{ab, cd\} \\ \mathscr{L}_2 &= \{ac, bd\} \\ \mathscr{L}_3 &= \{ad, bc\} \\ \mathscr{L}_4 &= \{ae, bf, cg, dh\} \\ \mathscr{L}_5 &= \{af, bg, ch, de\} \\ \mathscr{L}_6 &= \{ag, bh, ce, df\} \\ \mathscr{L}_7 &= \{ah, be, cf, dg\} \\ \mathscr{L}_8 &= \{ef, gh\} \\ \mathscr{L}_9 &= \{eg, fh\} \\ \mathscr{L}_{10} &= \{eh, fg\} \end{aligned}$$

The symbols can be partitioned into two subsets $\{a, b, c, d\}$ and $\{e, f, g, h\}$ which give this example a certain symmetry. This kind of partitioning of symbols will play a major role in the lower-bound constructions of chapter 5.

Proposition 3.2.5. *Let d, n, $\mathscr{F}$, and $\mathscr{V}$ be as in definition 3.2.1. Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a partition of $\mathscr{V}$. Then the following are equivalent:*

a) $\mathscr{L}$ is an abstract polyhedral graph.

b) Every set $u \subset \mathscr{F}$ of symbols is alive in a contiguous range of levels; this range may be empty.

Proof. **a) $\Longrightarrow$ b)** Suppose $u \subset \mathscr{F}$ is alive somewhere. Then let i be minimal such that u is alive in $\mathscr{L}_i$ and let k be maximal such that u is alive in $\mathscr{L}_k$. We need to show that u is alive in all levels in between. Take a vertex $v \in \mathscr{L}_i$, $u \subseteq v$ and a vertex $w \in \mathscr{L}_j$, $u \subseteq w$. Then by definition $v \cap w$ is alive in all levels between $\mathscr{L}_i$ and $\mathscr{L}_k$, but $u \subseteq (v \cap w)$, so u is alive as well.

b) $\Longrightarrow$ a) Let $1 \le i < j < k \le \ell$, $v \in \mathscr{L}_i$ and $w \in \mathscr{L}_k$. Clearly $(v \cap w) \subseteq v$, so $(v \cap w)$ is alive in $\mathscr{L}_i$. Similarly, $(v \cap w)$ is alive in $\mathscr{L}_k$. Since $(v \cap w)$ is alive in a contiguous range of levels, it must be alive in all levels between $\mathscr{L}_i$ and $\mathscr{L}_k$ and thus it must be alive in $\mathscr{L}_j$.

□

All upper bound proofs in this thesis are inductive proofs of some kind. The following two propositions are the tools that allow this kind of proof where the diameter of an abstract polyhedral graph is computed using the diameters of what one might call "subgraphs".

Proposition 3.2.6. *Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a (d,n)-abstract polyhedral graph. Let $1 \le j_1 < j_2 < \ldots < j_k \le \ell$ be a subsequence. Then $\mathscr{L}' = (\mathscr{L}_{j_1}, \ldots, \mathscr{L}_{j_k})$ is a (d,n)-abstract polyhedral graph.*

Proof. Going to subsequences keeps contiguous ranges contiguous, so the statement follows from proposition 3.2.5. □

The second proposition concerns itself with *induced* graphs of lower dimension. Recall that the intersection of facets of a polyhedron is a face, which is a polyhedron of lower dimension. Intuitively, taking the subgraph induced by a set of symbols is essentially the same operation as taking a face of a polyhedron.

Proposition 3.2.7. *Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a (d,n)-abstract polyhedral graph on the set $\mathscr{F}$ of symbols. Let $u \subset \mathscr{F}$ be a set of symbols that is alive somewhere in $\mathscr{L}$. Let $\mathscr{L}_a$ be the first and $\mathscr{L}_b$ be the last level on which u is alive. Then $\mathscr{L}' = (\mathscr{L}'_a, \ldots, \mathscr{L}'_b)$ with*

$$\mathscr{L}'_j = \{v \setminus u \mid v \in \mathscr{L}_j \text{ and } u \subset v\}$$

is a $(d - |u|, n - |u|)$-abstract polyhedral graph on the set $\mathscr{F} \setminus u$ of symbols.

If $\mathscr{L}$ is complete, then so is $\mathscr{L}'$.

Proof. Let $a \le i < j < k \le b$ and let $v \in \mathscr{L}'_i$ and $w \in \mathscr{L}'_k$. Then $(v \cup u) \in \mathscr{L}_i$ and $(w \cup u) \in \mathscr{L}_k$, so since $\mathscr{L}$ is an abstract polyhedral graph, there is an $x \in \mathscr{L}_j$ such that $(v \cup u) \cap (w \cup u) \subset x$. Using distributivity, $u \cup (v \cap w) \subset x$, so that $(x \setminus u) \in \mathscr{L}'_j$ and $(v \cap w) \subset (x \setminus u)$. Thus $\mathscr{L}'$ is an abstract polyhedral graph.

Note that the vertex set of $\mathscr{L}'$ is simply $\{v \setminus u \mid v \in \mathscr{V}\}$ if $\mathscr{V}$ is the vertex set of $\mathscr{L}$. Therefore, $\mathscr{L}'$ is complete if $\mathscr{L}$ is complete. □

3.3 Blueprints

It turns out that relaxing the definition of abstract polyhedral graphs to allow multisets of symbols as vertices is a valuable tool in the proof that complete abstract polyhedral graphs with large diameter exist. I call this further generalization *blueprint* due to its role in the construction in chapter 5.

Definition 3.3.1. *Let $1 \le d \le n$ be integers. Let $\mathscr{F}$ be a set of n symbols and let $\mathscr{V}$ be a family of d-element multisets consisting of symbols of $\mathscr{F}$. Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a* *partition of $\mathcal{V}$ into* levels. *Then $\mathcal{L}$ is a* (d,n)-blueprint *if the following connectedness condition holds: For all integers $1 \le i < j < k \le \ell$ and for all $u \in \mathcal{L}_i$, $v \in \mathcal{L}_k$, there exists a $w \in \mathcal{L}_j$ such that $(u \cap v) \subseteq w$.*

We say that the blueprint $\mathcal{L}$ has diameter $\operatorname{diam}(\mathcal{L}) := \ell - 1$. *The maximum diameter of a (d,n)-blueprint is denoted by $\Delta_{BP}(d,n)$.*

A blueprint is complete *if its vertex set $\mathcal{V}$ is the set $\left(\!\!\binom{\mathcal{F}}{d}\!\!\right)$ of all d-element multisets over symbols in $\mathcal{F}$. The maximum diameter of a complete (d,n)-blueprint is denoted by $\overline{\Delta_{BP}}(d,n)$.*

Example 3.3.2. The following is a $(3,3)$-blueprint over the symbols $\mathcal{F} = \{a,b,c\}$.

$$\begin{aligned} \mathcal{L}_1 &= \{aaa\} \\ \mathcal{L}_2 &= \{aab\} \\ \mathcal{L}_3 &= \{abb\} \\ \mathcal{L}_4 &= \{bbb\} \\ \mathcal{L}_5 &= \{bbc\} \\ \mathcal{L}_6 &= \{bcc\} \\ \mathcal{L}_7 &= \{ccc\} \end{aligned}$$

Example 3.3.3. Every abstract polyhedral graph is also a blueprint.

The following propositions are straightforward analogues of the corresponding propositions for abstract polyhedral graphs. I have omitted the proofs, since they are essentially identical to the corresponding proofs given in section 3.2. The only interesting difference is that in an induced abstract polyhedral graph, the symbols that the induction is on are removed permanently, while they can still appear in an induced blueprint.

Proposition 3.3.4. *Let d, n, $\mathcal{F}$, and $\mathcal{V}$ be as in definition 3.3.1. Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_\ell)$ be a partition of $\mathcal{V}$. Then the following are equivalent:*

a) $\mathcal{L}$ is a blueprint.

b) Every multiset u of symbols is alive in a contiguous range of levels; this range may be empty.

Proposition 3.3.5. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_\ell)$ be a (d,n)-blueprint. Let $1 \le j_1 < j_2 < \dots < j_k \le \ell$ be a subsequence. Then $\mathcal{L}' = (\mathcal{L}_{j_1}, \dots, \mathcal{L}_{j_k})$ is a (d,n)-blueprint.*

Proposition 3.3.6. *Let $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_\ell)$ be a (d,n)-blueprint on the set $\mathcal{F}$ of symbols. Let $u \subset \mathcal{F}$ be a multiset of symbols that is alive somewhere in $\mathcal{L}$. Let $\mathcal{L}_a$ be the first and $\mathcal{L}_b$ be the last level on which u is alive. Then $\mathcal{L}' = (\mathcal{L}'_a, \dots, \mathcal{L}'_b)$ with*

$$\mathcal{L}'_j = \{v \setminus u \mid v \in \mathcal{L}_j \text{ and } u \subset v\}$$

is a $(d - |u|, n)$-blueprint on the set $\mathcal{F}$ of symbols.

If $\mathcal{L}$ is complete, then so is $\mathcal{L}'$.

3.4 Hierarchy of abstractions

This section is devoted to the proof of the following theorem which establishes a hierarchy among convex polyhedra, convex polytopes, and the abstractions defined so far.

Theorem 3.4.1. *For all* $1 \le d \le n$*, the following holds:*

i) $\Delta_{AB}(d,n) \le \Delta_{AU}(d,n) \le \Delta_{UF}(d,n) \le \Delta_{APG}(d,n) \le \Delta_{BP}(d,n)$

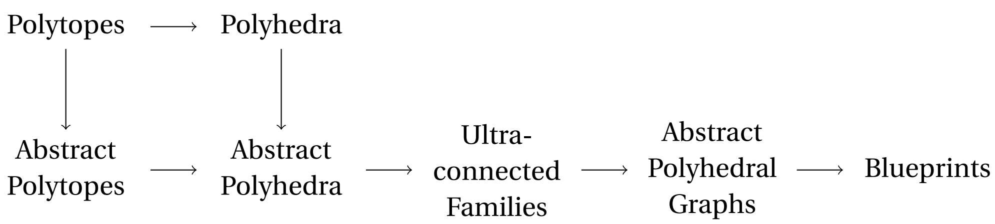
ii) $\Delta_B(d,n) \le \Delta_{AB}(d,n)$

iii) $\Delta_U(d,n) \le \Delta_{AU}(d,n)$

Proof. For the first two inequalities in i), see the simple proposition 3.1.6. The third inequality in i) follows from lemma 3.4.2. The last inequality follows from the fact that every abstract polyhedral graph is a blueprint of the same type and diameter.

Part ii) follows from lemmas 3.4.3 and 3.4.5: Let P be a d-dimensional polytope with n facets that has diameter $\Delta_B(d,n)$. Then there exists a simple polytope P' of the same type and diameter. This polytope induces an abstract polytope of the same diameter, which shows $\Delta_B(d,n) \le \Delta_{AB}(d,n)$. Part iii) follows in the same way. □

The following diagram visually illustrates the statement of theorem 3.4.1.



It is unclear which of the inequalities shown in this diagram are strict. What we do know for a fact is that for dimension $d \le 5$, the maximum diameter of APGs is strictly larger than the maximum diameter of abstract polytopes (this follows from [AD74] and theorem 5.4.2), and for dimension 4 and 5 the maximum diameter of polyhedra is strictly larger than the maximum diameter of polytopes.

Lemma 3.4.2. *Let* $\mathscr{V}$ *be an ultraconnected family of* d*-sets. Then there is a partition* $\mathscr{L} = (\mathscr{L}_0, \ldots, \mathscr{L}_\ell)$ *of* $\mathscr{V}$ *such that* $\mathscr{L}$ *is an abstract polyhedral graph and* $\operatorname{diam}(\mathscr{V}) = \operatorname{diam}(\mathscr{L}) = \ell$*.*

Proof. Let $v, w \in \mathscr{V}$ be two vertices such that $d(v,w) = \operatorname{diam}(\mathscr{V})$. Partition the vertices of $\mathscr{V}$ according to their distance from v, that is,

$$\mathscr{L}_j := \{u \in \mathscr{V} \mid d(u,v) = j\}$$

for all $0 \le j \le \ell = diam(V)$. Every vertex in $\mathscr{V}$ has a unique distance to v less than or equal to $\operatorname{diam}(\mathscr{V})$, so $\mathscr{L} = (\mathscr{L}_0, \ldots, \mathscr{L}_\ell)$ partitions $\mathscr{V}$.

Before we prove that $\mathscr{L}$ is an abstract polyhedral graph, note that whenever two vertices $u, u' \in \mathscr{V}$ are adjacent in $\mathscr{V}$, they must be either in the same or in adjacent levels in $\mathscr{L}$: Suppose, on the contrary, there were a pair of adjacent vertices $u, u' \in \mathscr{V}$ such that $u \in \mathscr{L}_j$ and $u' \in \mathscr{L}_k$ with $k > j+1$. Then $k = d(v, u') \leq d(v, u) + 1 = j + 1$ is clearly a contradiction.

Now let $0 \leq i < j < k \leq \ell$ indicate levels and let $u \in \mathscr{L}_i$ and $u' \in \mathscr{L}_k$. We need to show that there is a vertex $w \in \mathscr{L}_j$ such that $(u \cap u') \subseteq w$. By ultraconnectedness, there is a path from u to u' in $\mathscr{V}$ such that all vertices on the path are supersets of $u \cap u'$. By the previous argument, every step on the path either stays in a level or moves to an adjacent level, so that the path cannot skip a level. So there is a vertex w in the path that lies in level $\mathscr{L}_j$. □

The following lemma implies that there is a simple polyhedron (or polytope) that maximizes $\Delta_U(d, n)$ (or $\Delta_B(d, n)$). That is, it is enough to consider simple polyhedra (or polytopes) when talking about upper bounds on the diameter.

Lemma 3.4.3. *Let P be a (pointed) d-dimensional polyhedron with n facets. Then there exists a d-dimensional* simple *polyhedron P' with n facets and* $\operatorname{diam}(P) \leq \operatorname{diam}(P')$. *If P is a polytope, then P' is also a polytope.*

Proof. See [Kle64, 2.5]. □

Definition 3.4.4. *Let P be a simple polyhedron of dimension d. Let $\mathscr{F}$ be the set of facets of P. Then the family $\mathscr{V}$ of d-sets associated to P is defined as*

$$\mathscr{V} := \{\mathscr{F}[v] \mid v \text{ a vertex of } P\}$$

where $\mathscr{F}[v]$ is the set of facets that contain v.

Since P is simple, every vertex of P is contained in exactly d facets, so that calling $\mathscr{V}$ a family of d-sets is justified. Note that this definition is essentially the abstract simplicial boundary complex of the polar of P.

Lemma 3.4.5. *Let P be a simple polyhedron. Then the associated family $\mathscr{V}$ of d-sets is an abstract polyhedron and* $\operatorname{diam}(\mathscr{V}) = \operatorname{diam}(P)$. *Furthermore, if P is a polytope, then $\mathscr{V}$ is an abstract polytope.*

Proof. First of all, we will establish that the implied graph of $\mathscr{V}$ is naturally isomorphic to the graph of P by the isomorphism $v \mapsto F[v]$.

Let u and v be adjacent vertices of P. That is, there is an edge of P that contains u and v. This edge is the intersection of $d-1$ facets, all of which contain both u and v. Thus $|\mathscr{F}[u] \cap \mathscr{F}[v]| = d - 1$, i.e. u and v are adjacent in $\mathscr{V}$. The converse also holds, which shows that the graphs are isomorphic. In particular, this shows that the diameters are equal.

Now let $\mathscr{F}[u]$ and $\mathscr{F}[v]$ be two vertices in $\mathscr{V}$. Consider the facets in $\mathscr{F}[u] \cap \mathscr{F}[v]$. Their intersection with P defines a face P' of P. This face is a $(d - |F[u] \cap F[v]|)$-dimensional polyhedron whose graph is a subgraph of the graph of P. Furthermore, u and v are vertices of P', so there exists a path $u = v_0, \dots, v_k = v$ in the graph of P'. Consider the corresponding path in $\mathscr{V}$. Since all vertices v_j are contained in P', they are contained in all facets in $\mathscr{F}[u] \cap \mathscr{F}[v]$ and thus $(\mathscr{F}[u] \cap \mathscr{F}[v]) \subseteq \mathscr{F}[v_j]$ for all $0 \le j \le k$, i.e. $\mathscr{V}$ is ultraconnected.

Finally, let $F \subseteq \mathscr{F}$ be a set of $d-1$ facets. Note that the vertices in $\mathscr{V}$ that contain F are exactly the vertices of P that lie in the intersection $\bigcap F$ of the facets in F. So it remains to show that $\bigcap F$ contains at most two vertices for polyhedra, and exactly zero or two vertices for polytopes.

If $\bigcap F$ is empty, we are done. Otherwise, $\bigcap F$ is an edge of P. If the edge is unbounded, it contains exactly one vertex. Otherwise, it contains exactly two vertices. □

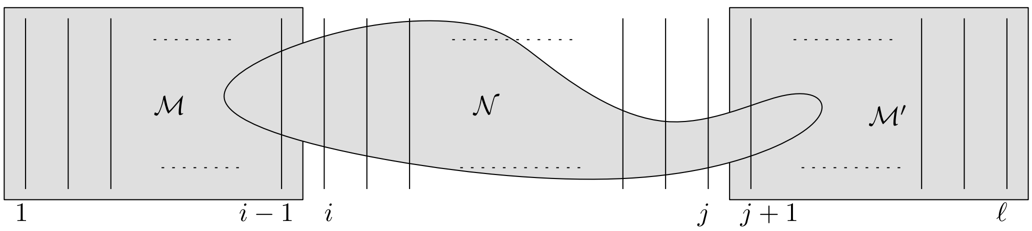
3.5 Upper bounds

Recall that the best known general upper bounds on the diameter of polyhedra are the bounds by Kalai and Kleitman [KK92] and by Larman [Lar70]. In this section, I will give proofs that these bounds hold even in the significantly more general framework of blueprints. In other words, very little of the geometry of the problem has to be used to obtain these bounds.

Theorem 3.5.1. $\Delta_{BP}(d,n) \le n^{\log d + 2}$.

Proof. We clearly have $\Delta_{BP}(1,n) \le n-1 < n^2$ because every level has to contain at least one vertex and there are only n possible vertices to choose from. So let $d > 1$ and suppose we have shown the statement for all lower dimensions.

Let $\mathscr{L} = (\mathscr{L}_1, \dots, \mathscr{L}_\ell)$ be a (d,n)-blueprint. Without loss of generality, all n symbols appear in the blueprint. Let $i \ge 1$ be the smallest integer such that more than $\frac{n}{2}$ symbols appear in the levels $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_i$ of the blueprint. I call these levels the *bottom part* of the blueprint. Similarly, let $j \le \ell$ be the largest integer such that more than $\frac{n}{2}$ symbols appear in the levels $\mathscr{L}_j, \mathscr{L}_{j+1}, \dots, \mathscr{L}_\ell$, which I call the *top part* of the blueprint (the following picture is rotated by 90 degrees).



My first claim is that $i - 2 \le \Delta_{BP}(d, \lfloor \frac{n}{2} \rfloor)$. This is obvious if $i = 1$. Otherwise, the levels $\mathscr{M} = (\mathscr{M}_1, \dots, \mathscr{M}_{i-1})$ form a blueprint by proposition 3.3.5. This blueprint has at most $\lfloor \frac{n}{2} \rfloor$ relevant symbols by the choice of i and its diameter is $i-2$. Therefore, $i - 2 \le \Delta_{BP}(d, \lfloor \frac{n}{2} \rfloor)$.

Similarly, one shows that $\ell - j - 1 \le \Delta_{BP}(d, \lfloor \frac{n}{2} \rfloor)$ using a blueprint $\mathscr{M}'$ formed by a subsequence of levels at the top.

My second claim is that $j-i \le \Delta_{BP}(d-1,n)$. By the pigeon hole principle, there has to be some symbol a that appears in both the bottom and the top parts of the blueprint. That is, the symbol a appears at least in levels $\mathscr{L}_i$ up to and including level $\mathscr{L}_j$. Now consider the $(d-1)$-dimensional blueprint $\mathscr{N}$ induced by a via proposition 3.3.6. Since this blueprint is taken from all those levels "in the middle", its diameter must be at least $j-i$. Therefore, $j-i \le \operatorname{diam}(\mathscr{N}) \le \Delta_{BP}(d-1,n)$.

Now we have $\operatorname{diam}(\mathscr{L}) = \ell - 1 \le 2(\Delta_{BP}(d, \lfloor \frac{n}{2} \rfloor)+1) + \Delta_{BP}(d-1,n)$. Since this holds for all blueprints of dimension at least 2, we can calculate recursively:

$$\begin{aligned}
\operatorname{diam}(\mathscr{L}) &\le 2(\Delta_{BP}(d, \left\lfloor \frac{n}{2} \right\rfloor)+1) + \Delta_{BP}(d-1,n) \\
&\le 2(\Delta_{BP}(d, \left\lfloor \frac{n}{2} \right\rfloor)+1) + 2(\Delta_{BP}(d-1, \left\lfloor \frac{n}{2} \right\rfloor)+1) + \Delta_{BP}(d-2,n) \\
&\le 2\sum_{k=2}^{d}(\Delta_{BP}(k, \left\lfloor \frac{n}{2} \right\rfloor)+1) + \Delta_{BP}(1,n) \\
&\le 2\sum_{k=2}^{d}\left(\left(\frac{n}{2}\right)^{\log k+2}+1\right) + n - 1 = \frac{n^2}{2}\sum_{k=2}^{d} k^{\log n - 1} + 2(d-1) + n - 1 \\
&\le \frac{n^2}{2}(d-1)d^{\log n-1} + 3(n-1) \le n^{\log d + 2}
\end{aligned}$$

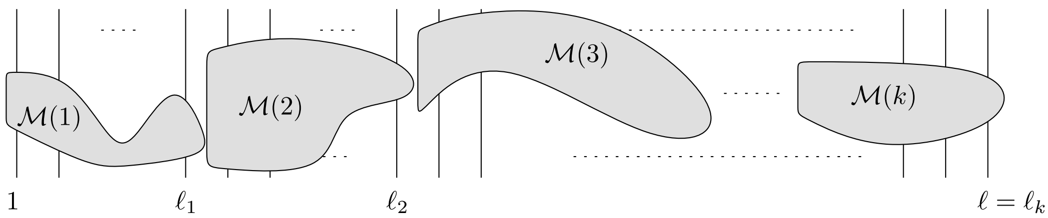
The calculation implicitly used $n \ge d \ge 2$ as well as induction on n for the upper bound of $\Delta_{BP}(d, \lfloor \frac{n}{2} \rfloor)$. Since we found an upper bound on the diameter of all (d,n)-blueprints, we get $\Delta_{BP}(d,n) \le n^{\log d+2}$. □

The second upper bound is worse in general. Nevertheless, it is interesting as it behaves linearly in the number of symbols for fixed dimension.

Theorem 3.5.2. $\Delta_{BP}(d,n) \le 2^{d-1}n - 1$.

Proof. For $d=1$, the statement is clear. Now let $d>1$ and assume that the statement is true for smaller dimension.

Let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ be a (d,n)-blueprint. Define integers $0 = \ell_0 < \ell_1 < \ldots < \ell_k = \ell$ recursively by choosing (for $j \ge 1$) ℓ_j as the largest integer such that there exists some symbol a_j that appears in both level $\mathscr{L}_{\ell_{j-1}+1}$ and in level $\mathscr{L}_{\ell_j}$ (and therefore in all levels in between). Let $\mathscr{M}(j)$ be the blueprint that is obtained by inducing on a_j according to proposition 3.3.6, and subsequently cutting off all levels (if any) that correspond to levels before $\mathscr{L}_{\ell_{j-1}+1}$ by proposition 3.3.5.



Let n_j be the number of symbols that appear in $\mathcal{M}(j)$. Since $\mathcal{M}(j)$ still contains levels corresponding to $\mathcal{L}_{\ell_{j-1}+1}$ up to and including $\mathcal{L}_{\ell_j}$, we have:

$$\ell_j - (\ell_{j-1}+1) \le \operatorname{diam}(\mathcal{M}(j)) \le \Delta_{BP}(d-1, n_j) \le 2^{d-2} n_j - 1$$

Combining this result for all blueprints $\mathcal{M}(j)$, we get:

$$\operatorname{diam}(\mathcal{L}) + 1 = \ell = \sum_{j=1}^{k} (\ell_j - \ell_{j-1}) = \sum_{j=1}^{k} (\ell_j - (\ell_{j+1}+1) - 1) \le \sum_{j=1}^{k} 2^{d-2} n_j$$

Now assume that there is some symbol b that appears in three or more of the $\mathcal{M}(j)$, and consider the range of levels on which b appears. Since the $\mathcal{M}(j)$ correspond to disjoint sequences of levels in $\mathcal{L}$, this range must overlap at least three ranges of type $[\ell_{j-1}+1, \ell_j]$. In particular, there is an i such that b appears on all levels in the range $[\ell_{i-1}+1, \ell_i+1]$. However, this contradicts the fact that ℓ_i was chosen to be maximal such that a symbol exists that appears on all levels in the range $[\ell_{i-1}+1, \ell_i]$.

Therefore, all symbols appear in at most two of the $\mathcal{M}(j)$, which means $\sum_{j=1}^{k} n_j \le 2n$. Putting it all together, we get $\operatorname{diam}(\mathcal{L}) \le \sum_{j=1}^{k} 2^{d-2} n_j - 1 \le 2^{d-1} n - 1$. □

While this is essentially the upper bound shown in [Lar70], the proof presented here is quite different from Larman's original proof. Larman showed that between any two vertices of a polyhedron there exists a path that does not revisit facets too often. A similar proof works for blueprints as well, but I believe the proof given here is easier to follow.

Also note that Larman's upper bound is actually $2^{d-3}n$, which is slightly better than the $2^{d-1}n$ shown here. He simply used the fact that the non-revisiting path conjecture is true in dimension $d \le 3$ and a similar trick could be used e.g. for abstract polytopes. Such a trick cannot work for blueprints or abstract polyhedral graphs as we will see in chapter 5. However, the important lesson learned here remains valid, which is that the techniques that allowed Larman to prove an upper bound for *all* dimensions can be used equally well for blueprints.

4 The disjoint coverings problem

In order to construct APGs of dimension d with large diameter, one has to figure out a way to populate each level with a small number of sets of size d such that a large number of sets of size less than d are *covered* by the d-sets. In this chapter, I will first present coverings of sets. Then I will extend the problem of finding small coverings to the disjoint coverings problem, that is the problem of finding a large family of disjoint coverings. The contributions of this chapter include two constructions that result in reasonable lower bounds for the general case and an asymptotically optimal result for a special case. This special case turns out to be exactly what we will need in chapter 5.

4.1 The covering problem

A well known problem in combinatorial design is the following:

Definition 4.1.1. *Let $0 \leq r < k < n$ be natural numbers and let X be a set of n symbols. An* (n,k,r)-covering (of X) *is a collection of k-sets of symbols such that every r-set of symbols is contained in at least one of the k-sets. I call such a collection simply* covering *if n, k and r are clear from the context. The* size *of an (n,k,r)-covering is the number of k-sets it contains. The covering number $C(n,k,r)$ is the minimum size of an (n,k,r)-covering.*

A 0-set is simply the empty set, so that for $r = 0$, a covering only needs to contain one arbitrary k-set.

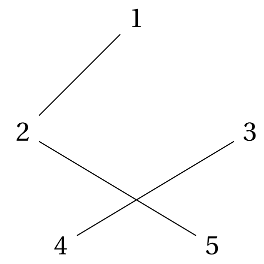
Remark 4.1.2. In the literature on combinatorial designs, coverings are usually endowed with an additional parameter λ indicating how many times each r-set must be covered. I chose to omit this parameter, always setting $\lambda = 1$ implicitly, so that it does not detract from the aspects which are of interest in this thesis.

Example 4.1.3. The following are two different $(8,3,1)$-coverings of the set $X = \{1,\ldots,8\}$:

$$\{123, 456, 678\}, \{123, 145, 167, 128\}$$

Example 4.1.4. Coverings can be understood in the language of hypergraphs. In the particularly simple case of $(n,2,1)$-coverings, the 2-sets can be understood as edges in an undirected graph that must cover the vertices. For example, consider the following $(5,2,1)$-covering and its interpretation as a set of edges in a graph with 5 vertices:

$$\{12, 34, 25\}$$



For a collection of results on coverings, see [CD06, chapter VI.11]. I will briefly cite some of the asymptotically interesting results. A simple counting argument shows that $C(n,k,r) \geq \binom{n}{r}/\binom{k}{r}$. The best known general lower bound for $C(n,k,r)$ is slightly larger:

$$C(n,k,r) \geq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \cdots \left\lceil \frac{n-r+1}{k-r+1} \right\rceil \cdots \right\rceil \right\rceil$$

It turns out that the natural lower bound is asymptotically tight. As Rödl [Rö85] first showed, we have for fixed k and r:

$$C(n,k,r) = \frac{\binom{n}{r}}{\binom{k}{r}}(1+o(1))$$

Moreover, there are efficient algorithms that find asymptotically optimal coverings, see e.g. [GPKS96].

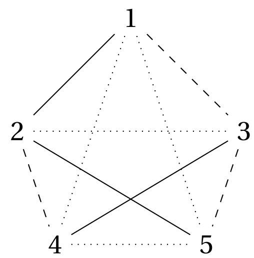
4.2 Families of disjoint coverings

Definition 4.2.1. *Let $0 \leq r < k < n$ be integers and let X be a set of n symbols. Let $\mathscr{C} = \{\mathscr{C}_1, \ldots, \mathscr{C}_s\}$ be a family of disjoint (n,k,r)-coverings. That is, each $\mathscr{C}_j$ is an (n,k,r)-covering and $\mathscr{C}_i \cap \mathscr{C}_j = \emptyset$ for all $i \neq j$. The* size *of $\mathscr{C}$ is the number $s = |\mathscr{C}|$ of coverings it contains. We say that $\mathscr{C}$ is* complete *if every k-set appears in exactly one of the coverings, that is if $\bigcup_{j=1}^{s} \mathscr{C}_j = \binom{X}{k}$.*

The disjoint covering number $DC(n,k,r)$ is the maximum size of a family of disjoint (n,k,r)-coverings.

Example 4.2.2. Complete families of disjoint coverings can be interpreted as colorings of the k-edges of a complete hypergraph, with the constraint that every r-edge is incident to at least one k-edge of every color. For example, the following is a complete family of disjoint $(5,2,1)$-coverings of size 3.

$$\{\{12,34,25\},\{13,24,35\},\{14,15,23,45\}\}$$



This section is devoted to the problem to determine the value of $DC(n,k,r)$ for given n, k, and r. Since finding an exact solution to this problem appears to be infeasible in the general case, we focus on lower and upper bounds.

Remark 4.2.3. The objects of previous study that seem to come closest are disjoint designs, e.g. [DS92, chapter 12], [Doy72], [Tei73]. These designs can be seen as coverings with additional constraints (from the perspective of coverings, one is tempted to call them *exact coverings*, in analogy to the NP-complete problem [GJ79, p. 221]). The results of these investigations into disjoint designs are not general enough for our purposes, but they can be applied to some special cases. To give a simple example, [Den74] implies that $DC(15,3,1)$ achieves the upper bound that is given by Theorem 4.2.4.

One might also be tempted to draw a parallel to coverings with parameter $\lambda > 1$ (see Remark 4.1.2). Unfortunately, such a covering would have to be partitioned into coverings with $\lambda = 1$, which is not possible in general. For example, consider the following $(4,3,1)$-covering which covers every 1-set at least twice:

$$\{123, 124, 234\}$$

It is clearly impossible to partition this set into two simple $(4,3,1)$-coverings.

Partitioning coverings in this way somewhat reminds me of the problem of resolvability, see e.g. [GGL95, p. 711], [And97]. However, resolutions always partition designs into smaller designs of type $r = 1$.

In summary, while there are certainly a number of objects that appear to be similar, to my knowledge the literature contains no reference to families of disjoint coverings as presented in this thesis.

Theorem 4.2.4. *For all integers* $0 \leq r < k < n$, $DC(n,k,r)$ *is bounded by*

$$\left\lfloor \binom{k}{r}^{-1} \cdot \binom{n-r}{k-r} \right\rfloor \leq DC(n,k,r) \leq \binom{n-r}{k-r}$$

Proof. To see the upper bound, fix an r-subset A. Every covering needs to contain at least one k-set B that contains A. However, there are only $\binom{n-r}{k-r}$ possible choices for B, so the size of a family of disjoint coverings is bounded by that number.

For the lower bound, we will construct a family of disjoint coverings of size

$$\ell := \left\lfloor \binom{k}{r}^{-1} \cdot \binom{n-r}{k-r} \right\rfloor$$

over the ground set $\mathbb{Z}_n$. Consider the bipartite graph $G = (V \cup W, E)$, where $V = \binom{\mathbb{Z}_n}{r} \times \mathbb{Z}_\ell$, $W = \binom{\mathbb{Z}_n}{k}$, and E contains an edge from $(A, i) \in V$ to $B \in W$ if and only if $A \subset B$. We will show that G satisfies the condition of the Marriage Theorem.

Let S be any subset of V, and let $N(S)$ denote the set of neighbors of vertices in S. We will count the number e_S of edges in the subgraph induced by $S \cup N(S)$.

Consider any vertex $(A, i) \in S$. Since A is an r-element set, there are $\binom{n-r}{k-r}$ ways to use the remaining elements of $\mathbb{Z}_n$ to fill A up to a block. Therefore, (A, i) has $\binom{n-r}{k-r}$ neighbors in G and in $N(S)$. We conclude

$$e_S = |S| \cdot \binom{n-r}{k-r}$$

Now consider any vertex $B \in N(S)$. There exist $\binom{k}{r}$ r-element subsets of B, and each of these gives rise to ℓ vertices in V that are neighbors of B in G. Thus the degree of B in G is exactly $\binom{k}{r} \cdot \ell \leq \binom{n-r}{k-r}$. Since the degree of B in a subgraph can only be smaller, we have

$$e_S \leq |N(S)| \cdot \binom{n-r}{k-r}$$

This gives us $|N(S)| \geq |S|$ and thus a V-perfect matching M exists in G by theorem 2.3.5. Now for every $j \in \mathbb{Z}_\ell$, we construct the covering C_j by collecting all blocks $B \in W$ that are matched to a vertex $(A, j) \in V$. Each C_j is a covering. In fact, if A is an r-subset of $\mathbb{Z}_n$, let $B \in W$ be the vertex that is matched to $(A, j) \in V$. Then $A \subset B$ and B is contained in the collection C_j. The collections $C_0, \ldots, C_{\ell-1}$ are disjoint since M matches each block to at most one vertex in V. Hence we have a family of ℓ disjoint coverings. □

There is a straightforward connection between the existence of large families of disjoint coverings and the existence of small coverings. The following proposition states that if there exists a family of disjoint coverings that is only by a factor of $\lambda \geq 1$ smaller than the natural maximum size from theorem 4.2.4, then there exists a covering that is only by a factor of λ bigger than the lower bound $C(n, k, r) \geq \binom{n}{r} / \binom{k}{r}$ stated in section 4.1.

Proposition 4.2.5. *Let $0 \leq r < k < n$, $\lambda \geq 1$, and suppose $DC(n, k, r) \geq \lambda^{-1}\binom{n-r}{k-r}$. Then $C(n, k, r) \leq \lambda \frac{\binom{n}{r}}{\binom{k}{r}}$.*

Proof. Let $\mathscr{C} = \{\mathscr{C}_1, \ldots, \mathscr{C}_s\}$ be a family of disjoint coverings of size $s \geq \lambda^{-1}\binom{n-r}{k-r}$. Since the coverings are disjoint, we have $\sum_{j=1}^{s} |\mathscr{C}_j| \leq \binom{n}{k}$. There is a $j \in \{1, \ldots, s\}$ such that

$$|\mathscr{C}_j| \leq \frac{\binom{n}{k}}{s} \leq \frac{\binom{n}{k}}{\lambda^{-1}\binom{n-r}{k-r}} = \lambda \frac{n!(k-r)!}{(n-r)!k!} = \lambda \frac{\binom{n}{r}}{\binom{k}{r}}$$

This particular $\mathscr{C}_j$ is a covering that shows the statement to be true. □

4.3 The case $k = r + 1$

We will now focus on the case $k = r + 1$. Theorem 4.3.1 shows a lower bound that is asymptotically equal to the upper bound established by theorem 4.2.4 in the sense that the quotient goes to 1 as n goes to infinity for any fixed r:

$$\frac{\binom{n-r}{k-r}}{n - r(r+2)} = \frac{n-r}{n - r(r+2)} \to 1$$

The first part of the proof of the following theorem was inspired by a method to obtain an optimal edge coloring of complete graphs found in [Iwa05], while the second part is another application of theorem 2.3.5, building upon the method used in the proof of theorem 4.2.4.

Theorem 4.3.1. *If $0 \le r < n$, then $DC(n, r+1, r) \ge n - r(r+2)$.*

Proof. If $r = 0$, the statement is trivially true since only one 1-set is needed for a covering. For the rest of the proof, I will assume $r \ge 1$.

Use $\mathbb{Z}_n$ as groundset and for any subset $A \subseteq \mathbb{Z}_n$ define $\sum(A) := \sum_{a \in A} a$. The function $b : \binom{\mathbb{Z}_n}{r+1} \to \mathbb{Z}_n$, $b(B) := \sum(B)$ assigns every block into one of the *buckets* $b^{-1}(\{0\})$, $b^{-1}(\{1\})$, up to $b^{-1}(\{n-1\})$. These buckets are disjoint. Unfortunately, they are no coverings. In fact, the following holds.

Lemma 4.3.2. *Every r-subset $A \subset \mathbb{Z}_n$ is covered in exactly $n - r$ buckets. Conversely, there exist exactly r buckets in which A is not covered by any block.*

Proof. Let A be an r-subset of $\mathbb{Z}_n$. Any block that covers A must be of the form $A \cup \{x\}$ with $x \in \mathbb{Z}_n \setminus A$, so there are exactly $n - r$ such blocks. They are sorted into the buckets with numbers $\hat{b}(x) = \sum(A) + x$. Since $\hat{b}$ is injective they are sorted into different buckets. □

From now on, the buckets numbered $0, \ldots, n - r(r+2) - 1$ will be called *surviving* buckets while the buckets numbered $n - r(r+2), \ldots, n-1$ will be called *supply* buckets. Using similar ideas as in theorem 4.2.4, we will fill up holes in the surviving buckets to turn them into coverings, using blocks from the supply buckets. Construct the bipartite graph $G = (V \cup W, E)$, where

$$\begin{aligned} V &:= \{(A, j) \mid A \in \tbinom{\mathbb{Z}_n}{r} \text{ is not covered in surviving bucket } j\} \\ W &:= \bigcup_{n - r(r+2) \le j < n} b^{-1}(j) \end{aligned}$$

Define E such that an edge between $(A, j) \in V$ and $B \in W$ exists if and only if $A \subset B$. We will show that G satisfies the conditions of theorem 2.3.5.

Let S be any subset of V. We will count the number e_S of edges in the subgraph induced by $S \cup N(S)$.

Consider any $B \in N(S)$. There are $r+1$ different r-element subsets of B. By the lemma, each of these fails to be covered in exactly r buckets. Some of these buckets may be supply buckets. Nevertheless, this means that B has at most degree $r(r+1)$ in G. Since the degree in a subgraph can only be smaller, we have

$$e_S \leq r(r+1)|N(S)|$$

Now consider any $(A, j) \in S$. Again by the lemma, there are at least $r(r+1)$ supply buckets that contain a superset of A. This means that (A, j) has at least degree $r(r+1)$ in G. Since we are looking at a subgraph that contains all neighbors of (A, j), the degree stays the same in the subgraph. Taking all vertices in S, we get

$$e_S \geq r(r+1)|S|$$

We conclude $|N(S)| \geq |S|$. By theorem 2.3.5, this shows the existence of a V-perfect matching in G. We add to each surviving bucket j all blocks $B \in W$ such that B is matched up with some (A, j). Since each $B \in W$ ends up in at most one surviving bucket, the surviving buckets remain disjoint, and by the construction, they become coverings. □

4.4 Improving the general case

In the following chapters, I will only need the special case dealt with in the previous section. Being curious, however, I did investigate the general case of families of disjoint coverings as well.

Theorem 4.4.1. *Let $r \geq 1$ and $d \geq 1$ be fixed integers. Then $DC(n, r+d, r) \geq f(n, r+d, r)$, where $f(n, r+d, r) = \left(\frac{n}{d}\right)^d - \mathcal{O}(n^{d-1})$. The $\mathcal{O}$-term contains constants that depend on both d and r.*

Before we embark on the proof, note that the upper bound from theorem 4.2.4 is $DC(n, r+d, r) \leq \binom{n-r}{d}$. So the quotient of this upper bound and the lower bound from this section approaches the constant $\frac{d^d}{d!}$ as n goes to infinity.

On the other hand, the quotient between the upper bound and lower bound in theorem 4.2.4 approaches $\binom{r+d}{d} = \frac{(r+d)\cdots(r+1)}{d!}$. Thus, the lower bound shown in this section is a significant improvement when r is large. It is worse than the naive lower bound when r is small. This is no surprise considering that the proof below is quite wasteful in this case.

Proof of Theorem 4.4.1. Let n be a sufficiently large integer. Let X be a set of n symbols. Partition X into sets $X_1, \ldots, X_d$ such that each of the sets contains at least $\lfloor \frac{n}{d} \rfloor$ symbols. The basic idea of the proof is to find families of disjoint $(n, q+1, q)$-coverings for each of the X_i and to combine those into a family of disjoint $(n, r+d, r)$-covering of X.

For every $1 \leq i \leq d$ and for every $0 \leq q \leq r$, let $X_i^q = \{X_i^q(0), \ldots, X_i^q(m-1)\}$ be families of disjoint $(n, q+1, q)$-coverings of X_i, where m is the largest natural number such that such X_i^q exist.

Let $F := \{1, \dots, d\}$ and let $V := \left(\!\!\binom{F}{r}\!\!\right)$ be the set of r-element multisets over F. Now for every vector $j \in (\mathbb{Z}_m)^d$ construct the $(n, r+d, r)$-covering C_j as follows:

$$C_j := \bigcup_{v \in V} \bigotimes_{i=1}^{d} X_i^{v_i}(j_i)$$

What does this definition mean? First of all, all elements of $X_i^{v_i}(j_i)$ are (v_i+1)-sets with respect to X_i, and therefore all elements of the unordered product $\bigotimes_{i=1}^{d} X_i^{v_i}(j_i)$ are $(v_1 + 1, \dots, v_d + 1)$-sets with respect to $(X_1, \dots, X_d)$ so that all elements of C_j are $(r+d)$-sets with respect to X. The converse is also true, that is every set $M \in C_j$ is a $(v_1+1, \dots, v_d+1)$-set which comes from the unordered product corresponding to v.

I still have to show that C_j is, in fact, an $(n, r+d, r)$-covering of X. Let M be an arbitrary r-set with respect to X. Then M is a v-set with respect to $(X_1, \dots, X_d)$ for some $v \in V$. For every $1 \le i \le d$, let M_i be the X_i-component of M, i.e. $M_i = M \cap X_i$ and $|M_i| = v_i$. By our initial choice of families of disjoint coverings there is an $N_i \in X_i^{v_i}(j_i)$ that covers M_i, that is $N_i \supset M_i$. Moreover, $N = N_1 \cup \ldots \cup N_d$ covers M and is an element of the unordered product $\bigotimes_{i=1}^{d} X_i^{v_i}(j_i)$. Therefore, $N \in C_j$. This establishes that C_j is a covering.

Now let $j, j' \in (\mathbb{Z}_m)^d$ with $j \ne j'$. I want to show that C_j and $C_{j'}$ are disjoint, so assume they are not disjoint. Then there is some $v \in V$ and some $(v_1+1, \dots, v_d+1)$-set $M \in C_j \cap C_{j'}$. Let $1 \le i \le d$ be a position such that $j_i \ne j'_i$. Let M_i be the X_i-component of M. As discussed above, $M_i \in X_i^{v_i}(j_i)$ and $M_i \in X_i^{v_i}(j'_i)$. So we found two coverings in $X_i^{v_i}$ that are not disjoint, which is a contradiction. This establishes that

$$C := \{C_j \mid j \in (\mathbb{Z}_m)^d\}$$

is a family of disjoint $(n, r+d, r)$-coverings of size m^d.

By theorem 4.3.1, $m \ge \lfloor \frac{n}{d} \rfloor - r(r+2)$.

$$f(n,k,r) = \left(\frac{n}{d} - r(r+2) - 1\right)^d = \left(\frac{n}{d}\right)^d - \mathcal{O}(n^{d-1})$$

With that, $DC(n,k,r) \ge m^d \ge f(n,k,r)$. □

5 Lower bounds for the diameter problem

In this chapter, I construct an infinite family of abstract polyhedral graphs of diameter close to dn. While it remains possible that a polynomial upper bound for the diameter of polyhedra can be proved for abstract polyhedral graphs, this result proves that the structure encoded in abstract polyhedral graphs is not sufficient for settling the Hirsch conjecture. If one tries to prove the Hirsch conjecture, one will have to utilize more aspects of the geometry of convex polytopes.

One can easily construct blueprints of diameter $d(n-1)$. In fact, all that is required is a simple generalization of example 3.3.2. The majority of this chapter is devoted to a reduction that, given an arbitrary blueprint, constructs a related abstract polyhedral graph. In this way, I obtain a good lower bound on the maximum diameter of abstract polyhedral graphs.

5.1 The big picture

Throughout this chapter, let $\mathscr{L}$ be a (d,n)-blueprint over the set $\mathscr{F} = \{1,\ldots,n\}$ of symbols. Let $\mathscr{V}$ be the set of vertices in $\mathscr{L}$.

The central idea of the reduction is to use the vertices of the blueprint as patterns or templates for vertices in an abstract polyhedral graph. Every vertex of the blueprint is replaced by an entire *component*, a term which I will make precise very soon. For now, just think of it as a small APG with additional constraints.

The crucial point is that each vertex v of the blueprint determines the type of vertices in the corresponding component.[1] In particular, every vertex of the component is obtained by replacing each symbol $a \in v$ by an appropriate symbol out of a set $\mathscr{G}_a$. The resulting abstract polyhedral graph will then use the set

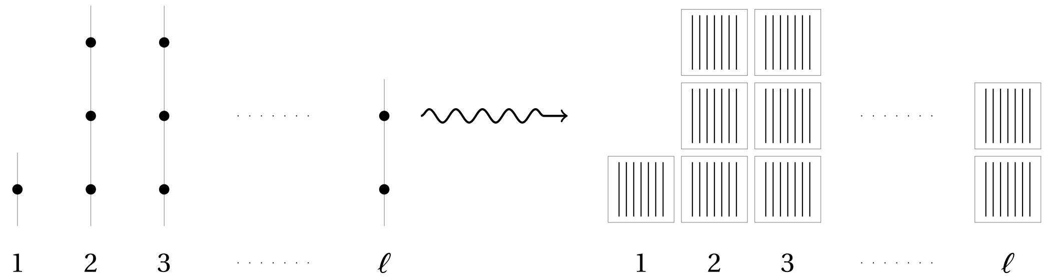
$$\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2 \cup \ldots \cup \mathscr{G}_n$$

of symbols, where the $\mathscr{G}_j$ are pairwise disjoint and $|\mathscr{G}_j| = m$ for a sufficiently large natural number m. I will use this notation consistently throughout the construction.

Since the number of symbols increases from n to nm, we lose some of the quality of the blueprint. However, we can recover most of that loss, because the diameter of the resulting abstract polyhedral graph will be larger by a factor close to m.

[1] This is where the term "blueprint" comes from. As mentioned earlier, the idea for blueprints arose while working on the construction of complete abstract polyhedral graphs.

The following picture visualizes the construction. Each vertex of the blueprint, i.e. each point on the left hand side, is replaced by a component with many levels, i.e. one small rectangle on the right hand side.



I will define and construct the components (the little rectangles) in section 5.2 and then put them together for the full construction in section 5.3.

5.2 Components

Recall the projection map $\pi : 2^{\mathscr{G}} \to \mathbb{N}^{\mathscr{F}}$ from section 2.2 that maps elements from $\mathscr{G}_a$ to a. For example, if $a, b \in \mathscr{G}_1$ and $c \in \mathscr{G}_3$, then $\pi(abc) = 113$, where 113 is a multiset of cardinality 3.

Definition 5.2.1. *Let $v \in \mathscr{V}$ be a vertex of the blueprint. Let $\mathscr{M} = (\mathscr{M}_1, \ldots, \mathscr{M}_l)$ be a d-dimensional APG over the symbols $\mathscr{G}$. Then $\mathscr{M}$ is a v-*component *$\mathscr{C}(v)$ if the following conditions hold:*

a) The set of all vertices of $\mathscr{M}$ is the set of all v-sets, that is $\bigcup_{j=1}^{l} \mathscr{M}_j = \pi^{-1}(\{v\})$.

b) For every $u \subsetneq v$, every u-set is covered on every level $\mathscr{M}_j$.

Note that the fact that a $\mathscr{C}(v)$-component is an abstract polyhedral graph already follows from the two last conditions by proposition 3.2.5.

Example 5.2.2. Suppose $\mathscr{G}_1 = \{a, b, c, d\}$ and $\mathscr{G}_2 = \{e, f, g, h\}$. Then we can construct a $\mathscr{C}(112)$-component in the following way. Take a complete family of disjoint $(4,2,1)$-coverings of $\mathscr{G}_1$, say

$$\{ab, cd\}, \{ac, bd\}, \{ad, bc\}$$

and a complete family of disjoint $(4,1,0)$-coverings of $\mathscr{G}_2$, say

$$\{e\}, \{f\}, \{g, h\}$$

Taking the following unordered products of sets

$$\begin{aligned}
\mathscr{M}_1 &= \{ab, cd\} \otimes \{e\} \cup \{ac, bd\} \otimes \{g, h\} \cup \{ad, bc\} \otimes \{f\} \\
\mathscr{M}_2 &= \{ab, cd\} \otimes \{f\} \cup \{ac, bd\} \otimes \{e\} \cup \{ad, bc\} \otimes \{g, h\} \\
\mathscr{M}_3 &= \{ab, cd\} \otimes \{g, h\} \cup \{ac, bd\} \otimes \{f\} \cup \{ad, bc\} \otimes \{e\}
\end{aligned}$$

we get:

$$\begin{aligned}\mathscr{M}_1 &= \{abe, cde, acg, bdg, ach, bdh, adf, bcf\}\\ \mathscr{M}_2 &= \{abf, cdf, ace, bde, adg, bcg, adh, bch\}\\ \mathscr{M}_3 &= \{abg, cdg, abh, cdh, acf, bdf, ade, bce\}\end{aligned}$$

The given construction is a special case of the proof of lemma 5.2.3, so we can be assured that the resulting sequence of levels $\mathscr{M}$ is, in fact, a $\mathscr{C}(112)$-component. Of course one could also verify this fact manually, but that gets rather tedious.

In the remainder of this section, I will prove that there exist $\mathscr{C}(v)$-components with a diameter that is not too small. The proof contains the core idea of how one can go from multisets, where a symbol can appear arbitrarily often, to sets, where a symbol cannot appear more than once. At this meeting point of different worlds, it turns out that it is valuable to think and talk about multisets in terms of their characteristic vector. In other words, a vertex $v \in \mathscr{V}$, while being a multiset, is at the same time a vector with n components, where each component is a nonnegative integer indicating how often the corresponding symbol appears in v.

Lemma 5.2.3. *For every $v \in \mathscr{V}$, there is a $\mathscr{C}(v)$-component with at least $m-(d^2-1)$ levels.*

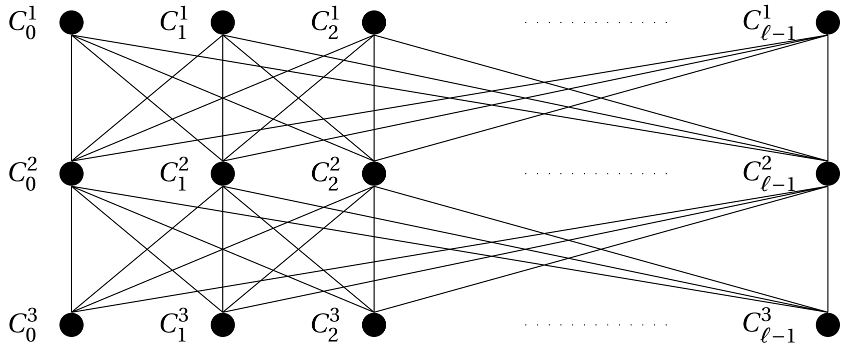
Proof. Without loss of generality, we can assume that all symbols of $\mathscr{F}$ appear at least once in v. We can simply ignore symbols (and their corresponding $\mathscr{G}$-sets) that do not occur in v. Equivalently, all components of v are nonzero when v is understood as an integer vector.

For every $1 \le j \le n$, choose a complete family of disjoint $(m, v_j, v_j - 1)$-coverings $C_0^j, \ldots, C_{\ell-1}^j$ of $\mathscr{G}_j$. Here we choose ℓ to be the largest number such that choosing families of disjoint coverings of size ℓ is possible for all j. By theorem 4.3.1, $\ell \ge m-(k^2-1) \ge m-(d^2-1)$, where k is the largest entry of v.

Let $\mathbb{Z}_\ell$ denote the integers modulo ℓ. For all $I \in (\mathbb{Z}_\ell)^n$, define

$$\begin{aligned}\sum(I) &:= \sum_{j=1}^{n} I_j\\ C_I &:= \bigotimes_{j=1}^{n} C_{I_j}^j\end{aligned}$$

What this amounts to is that C_I is the set of all v-sets where for all j the j-th component comes from the covering $C_{I_j}^j$. These v-sets will end up being the vertices of the component. Another way to look at this definition is to say that the entries of vector I describe the choice of coverings (one covering from each family of disjoint coverings) that are multiplied together to form the set C_I. The following diagram attempts to illustrate the situation for $n = 3$.



Each node in row j is an $(m, v_j, v_j - 1)$-covering of $\mathcal{G}_j$. The definition of C_I can be understood as walking along the path described by I, choosing an arbitrary v_j-element set at each node. The union of these sets becomes one of the vertices in C_I.

For all $j \in \mathbb{Z}_\ell$, let

$$\mathcal{M}_j := \bigcup_{I \in (\mathbb{Z}_\ell)^n, \sum(I)=j} C_I$$

In other words, the vertices in $\mathcal{M}_j$ are the v-sets such that when we describe the coverings from which the symbols of the vertex are taken by a vector $I \in (\mathbb{Z}_\ell)^n$ as above, then $\sum(I) = j$.

It remains to show that $(\mathcal{M}_0, \ldots, \mathcal{M}_{\ell-1})$ is a $\mathcal{C}(v)$-component. First of all, consider the set of vertices of the component:

$$\bigcup_{j \in \mathbb{Z}_\ell} \mathcal{M}_j \quad = \quad \bigcup_{I \in (\mathbb{Z}_\ell)^n} C_I = \bigcup_{I \in (\mathbb{Z}_\ell)^n} \bigotimes_{j=1}^{n} C_{I_j}^j = \bigotimes_{j=1}^{n} \bigcup_{i \in \mathbb{Z}_\ell} C_i^j = \bigotimes_{j=1}^{n} \binom{\mathcal{G}_j}{v_j}$$

So the set of vertices is the set of all v-sets. For the last equality, we used the fact that the families C^j of disjoint coverings are complete.

It remains to show that every u-set with $u \subsetneq v$ is covered in every $\mathcal{M}_j$. So let x be a u-set, $u \subsetneq v$, and let $j \in \mathbb{Z}_\ell$. We need to show that there is a $y \in \mathcal{M}_j$ such that $x \subset y$.

Choose $1 \leq \alpha \leq n$ such that $u_\alpha < v_\alpha$. Such an α exists because $u \subsetneq v$. For all $1 \leq \beta \leq n$, $\beta \neq \alpha$, the $\mathcal{G}_\beta$-component of x is contained in a set $y_\beta \in C_{I_\beta}^\beta$ for some I_β. To see that this is true, use arbitrary symbols from $\mathcal{G}_\beta$ to extend the $\mathcal{G}_\beta$-component of x to a v_β-subset $y_\beta \subseteq \mathcal{G}_\beta$. Then y_β is an element of one of the coverings $C_0^\beta, \ldots, C_{\ell-1}^\beta$ since the families of disjoint coverings we chose are complete.

Given those I_β ($\beta \neq \alpha$), choose $I_\alpha \in \mathbb{Z}_\ell$ such that $\sum(I) = j$. Since $C_{I_\alpha}^\alpha$ is an $(m, v_\alpha, v_\alpha - 1)$-covering and the $\mathcal{G}_\alpha$-component of x has fewer than v_α elements, there is a $y_\alpha \in C_{I_\alpha}^\alpha$ that contains the $\mathcal{G}_\alpha$-component of x. Putting it all together, we have

$$x \subseteq \bigcup_{\gamma=1}^{n} y_\gamma \in C_I \subseteq \mathcal{M}_j$$

In other words, x is covered by the vertex $y = \bigcup_{\gamma=1}^{n} y_\gamma$ which appears in the level $\mathcal{M}_j$. $\square$

The last step of the proof can be visualized as finding paths through the picture in the following way: Fix the path through all rows except for one. There remains one row on which we can choose the path freely. This allows us to find a path that will work for every level $j \in \mathbb{Z}_\ell$.

Finally, note that one can easily obtain $\mathscr{C}(v)$-components with fewer levels, simply by collapsing two or more levels into a single level.

5.3 Putting components together

Theorem 5.3.1. *Let $\mathscr{L} = (\mathscr{L}_0, \dots, \mathscr{L}_{\ell-1})$ be a (d,n)-blueprint. There exists a (d,nm)-abstract polyhedral graph $\mathscr{M}$ such that*

$$\operatorname{diam}(\mathscr{M}) \geq (m - d^2 + 1)(\operatorname{diam}(\mathscr{L}) + 1) - 1$$

If $\mathscr{L}$ is a complete blueprint, then $\mathscr{M}$ is a complete abstract polyhedral graph.

Proof. The picture in section 5.1 really describes the gist of the construction of $\mathscr{M}$. Keep this in mind while following the formal construction given here.

Let $\mathscr{V}$ be the set of vertices of the blueprint $\mathscr{L}$. For every $v \in \mathscr{V}$, let

$$\mathscr{C}(v) = (\mathscr{C}(v)_0, \dots, \mathscr{C}(v)_{m-d^2})$$

be a $\mathscr{C}(v)$-component. Lemma 5.2.3 guarantuees the existence of the $\mathscr{C}(v)$. Note that the vertex sets of the $\mathscr{C}(v)$ are disjoint. Define

$$\mathscr{M} := (\mathscr{M}_0, \dots, \mathscr{M}_{\ell(m-d^2+1)-1})$$

as follows. For $\mathscr{M}_j$, write $j = i(m - d^2 + 1) + k$ such that $0 \leq k < m - d^2 + 1$. Note that $0 \leq i < \ell$. Let

$$\mathscr{M}_j := \bigcup_{v \in \mathscr{L}_i} \mathscr{C}(v)_k$$

Now $\mathscr{M}$ has the right number of levels, and it is clear that if $\mathscr{L}$ is complete then $\mathscr{M}$ is complete. The $\mathscr{M}_j$ are disjoint because all levels of all components $\mathscr{C}(v)$ are mutually disjoint and each level appears only once in one of the unions $\mathscr{M}_0, \dots, \mathscr{M}_{\ell(m-d^2+1)-1}$. It remains to show that $\mathscr{M}$ is an abstract polyhedral graph. By proposition 3.2.5, it is enough to show that all sets of symbols are alive in a contiguous range of levels.

Let $x \subset \mathscr{G}$ be a set of symbols that is alive somewhere in $\mathscr{M}$. If $|x| = d$, x is a vertex and can therefore be alive on only one level. Consider the case $|x| < d$. We have that x is a u-set, where u is a multiset $u \subset \mathscr{F}$ and $|u| = |x| < d$.

Suppose x is alive on level $\mathscr{M}_j$ with $j = i(m - d^2 + 1) + k$ as above. Then x is alive in $\mathscr{C}(v)_k$ for some $v \in \mathscr{L}_i$ with $u \subset v$. That is, u is alive in level $\mathscr{L}_i$ of the blueprint.

Conversely, suppose u is alive in level $\mathscr{L}_i$ of the blueprint. Then there is a $v \in \mathscr{L}_i$ with $u \subsetneq v$. By the definition of components, this means that x is alive in every level of $\mathscr{C}(v)$. It follows that x is alive in the levels $\mathscr{M}_{i(m-d^2+1)}, \dots, \mathscr{M}_{(i+1)(m-d^2+1)-1}$.

So the set of symbols x is alive in ranges of type $[i(m-d^2+1),(i+1)(m-d^2+1)-1]$, and there is a one-one correspondence between those ranges and the levels in which u is alive in the blueprint. By proposition 3.3.4, u is alive in a contiguous range of levels of the blueprints, so x is alive in a contiguous range of levels of the abstract polyhedral graph. □

5.4 Lower bounds

The following lower bound on the maximal diameter of blueprints can be obtained simply by generalizing example 3.3.2. However, I will provide a better result which gives the same lower bound for *complete* blueprints in theorem 6.2.2. Therefore, I do not want to go into unnecessary details at this point.

Proposition 5.4.1. *For all* $1 \le d \le n$*:* $\Delta_{BP}(d,n) \ge d(n-1)$.

The main result in this chapter is a superlinear lower bound on the maximal diameter of abstract polyhedral graphs. For example, by letting $d = n^{\frac{1}{3}}$ in theorem 5.4.2, one obtains $\Omega(n^{\frac{4}{3}})$ as a lower bound.

Theorem 5.4.2. *For every* $d \ge 1$ *there exist infinitely many* n *with* $\Delta_{APG}(d,n) \ge dn - 2d^2\sqrt{n}$.

Proof. Let $k \ge 1$ be an integer and construct a blueprint of diameter $d(k-1)$ by proposition 5.4.1. Then use this blueprint in theorem 5.3.1 to construct an abstract polyhedral graph $\mathscr{L}$ with $|\mathscr{G}_j| = m = kd^2$ symbols per group. This abstract polyhedral graph has $n = km = k^2d^2$ symbols and its diameter is

$$\begin{aligned}
\operatorname{diam}(\mathscr{L}) &\ge (m-d^2+1)(d(k-1)+1)-1 \\
&= (kd^2-d^2+1)(dk-d+1)-1 \\
&= d^2(k-1)(dk-d+1)+(dk-d+1)-1 \\
&= d^2(dk^2-dk-dk+d+k-1)+dk-d \\
&= dn-2d^3k+d^3+d^2k-d^2+dk-d \\
&\ge dn-2d^2\sqrt{n}
\end{aligned}$$

This completes the proof. □

6 Complete blueprints and complete abstract polyhedral graphs

When trying to find polytopes, abstract polyhedral graphs, or blueprints with a large diameter, it is tempting to look among those instances with the largest number of vertices. Kalai [Kal91] showed that the diameter of polytopes with a maximal number of vertices is bounded by a polynomial. My results in this chapter include an analogue statement for abstract polyhedral graphs and blueprints. To be more precise, the maximal diameter of complete (d,n)-blueprints is exactly $d(n-1)$, while the maximal diameter of complete (d,n)-abstract polyhedral graphs is asymptotically equal to its upper bound of $d(n-d)$.

In other words, the diameter problem for this special case is essentially solved. If one hopes to find examples of abstract polyhedral graphs or blueprints with (asymptotically) larger diameter than the ones in this and the previous chapter, one must abandon the idea of maximal vertex sets.

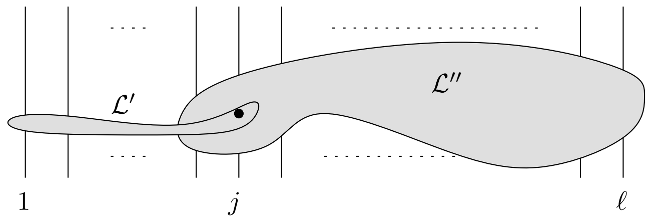
6.1 Polynomial upper bounds

Proposition 6.1.1. *For every complete (d,n)-abstract polyhedral graph $\mathscr{L}$ the diameter is bounded by* $\operatorname{diam}(\mathscr{L}) \le d(n-d)$*. Equivalently,* $\overline{\Delta_{APG}}(d,n) \le d(n-d)$.

Proof. The proof is by induction on d. For $d=1$, clearly every APG has at most n levels and therefore at most diameter $n-1$.

Now let $d>1$ and let $\mathscr{L}=(\mathscr{L}_1,\dots,\mathscr{L}_\ell)$ be a complete d-dimensional APG with n symbols. Let u be a $(d-1)$-set that is alive in the first level $\mathscr{L}_1$ and let x be a symbol that is alive in the last level $\mathscr{L}_\ell$. If $x\in u$, the diameter of $\mathscr{L}$ is already equal to the diameter of the complete $(d-1)$-dimensional APG that is induced by $\{x\}$ and therefore $\operatorname{diam}(L) \le \overline{\Delta_{APG}}(d-1,n-1) \le (d-1)(n-1-(d-1)) \le d(n-d)$ by the induction hypothesis.

Now consider the case $x\notin u$. Since $\mathscr{L}$ is complete, there is a vertex $u\cup\{x\}$ in one of the levels, say in $\mathscr{L}_j$. Now consider the 1-dimensional APG $\mathscr{L}'$ induced by u and the $(d-1)$-dimensional APG $\mathscr{L}''$ induced by $\{x\}$. Both of these APGs are complete by proposition 3.2.7. The APG $\mathscr{L}'$ spans at least the levels from $\mathscr{L}_1$ to $\mathscr{L}_j$, and $\mathscr{L}''$ spans at least the levels from $\mathscr{L}_j$ to $\mathscr{L}_\ell$. The following picture shows the situation:



Note that induced abstract polyhedral graphs have fewer symbols than the original abstract polyhedral graph because the symbols that we induce on are removed. In particular, $\mathscr{L}'$ has only $n-(d-1)$ symbols while $\mathscr{L}''$ has $n-1$ symbols. We can now calculate:

$$\begin{aligned}\operatorname{diam}(\mathscr{L}) &\leq \operatorname{diam}(\mathscr{L}') + \operatorname{diam}(\mathscr{L}'') \\ &\leq \overline{\Delta_{APG}}(1, n-(d-1)) + \overline{\Delta_{APG}}(d-1, n-1) \\ &\leq n-(d-1)-1+(d-1)(n-1-(d-1)) = n-d+(d-1)(n-d) \\ &= d(n-d)\end{aligned}$$

This holds for all complete d-dimensional APGs with n symbols, so $\overline{\Delta_{APG}}(d,n) \leq d(n-d)$. □

Proposition 6.1.2. *For every complete* (d,n)*-blueprint* $\mathscr{L}$*, we have* $\operatorname{diam}(\mathscr{L}) \leq d(n-1)$*. Equivalently,* $\overline{\Delta_{BP}}(d,n) \leq d(n-1)$.

Proof. The proof is almost identical to the proof of proposition 6.1.1. However, care must be taken because the number of symbols in induced blueprints is not reduced. Therefore, we only have $\operatorname{diam}(\mathscr{L}') \leq \overline{\Delta_{BP}}(1,n)$ and $\operatorname{diam}(\mathscr{L}'') \leq \overline{\Delta_{BP}}(d-1,n)$ and thus:

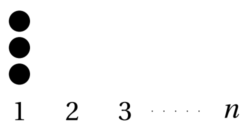
$$\operatorname{diam}(\mathscr{L}) \leq \overline{\Delta_{BP}}(1,n) + \overline{\Delta_{BP}}(d-1,n) \leq n-1+(d-1)(n-1) = d(n-1)$$

□

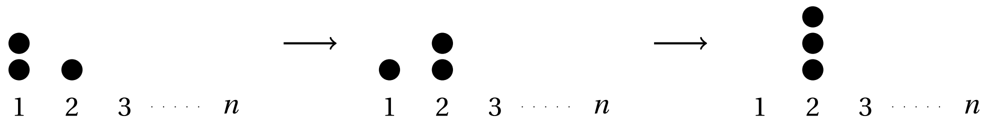
6.2 Lower bounds

The goal of this section is to construct complete blueprints with maximal diameter. Consider the blueprint in example 3.3.2. In a generalization of this blueprint over the set of symbols $\mathscr{F} = \{1,\ldots,n\}$, we start with a vertex of all 1s. Then one by one, each of the 1s gets replaced by a 2, until we end up with a vertex of all 2s. Then the 2s are replaced by 3s, and so on.

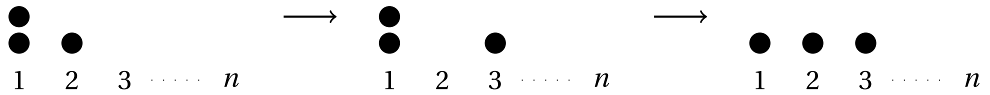
Think of it as a simple game with n positions and d chips. At the start of the game, all chips are in position 1, corresponding to the vertex 111 on the first level of the blueprint (in the illustration, $d=3$):



The only rule of the game is that we can take any chip we like and move it to the next position while moving to the next level of the blueprint at the same time. In the generalization of example 3.3.2, one performs moves like this:



This sequence of moves gives us the vertex 112 in the second level, 122 in the third level, and so on. One can continue this sequence in a natural way until all chips are in position n. Of course, other sequences of moves are possible, for example:



This gives us the vertex 113 in the third level and the vertex 123 in the fourth level. The idea behind the proof of theorem 6.2.2 is to allow all possible sequences of moves in this game and use this to generate the levels of a blueprints, except that I will label the symbols and positions from 0 to $n-1$ as this makes the formulas more natural.

The following potential function will help in describing this idea formally. Let $\mathscr{F} = \{0,\ldots,n-1\}$ be a set of symbols and denote with $\mathbb{N}^{\mathscr{F}}$ the set of multisets consisting of symbols of $\mathscr{F}$. Then define

$$\begin{aligned} \phi : \mathbb{N}^{\mathscr{F}} &\to \mathbb{N} \\ v &\mapsto \sum_{a \in v} a \end{aligned}$$

Intuitively, $\phi(v)$ describes the number of game moves that are necessary to go from the initial position to position v, so I will put the vector v into level $\phi(v)$. Note that because we are operating on multisets, ϕ is a homomorphism of monoids, that is $\phi(u \cup v) = \phi(u) + \phi(v)$ and $\phi(\emptyset) = 0$.

Lemma 6.2.1. $\phi(\left(\!\!\binom{\mathscr{F}}{d}\!\!\right)) = \{0,1,\ldots,d(n-1)\}$.

Proof. Clearly, ϕ is nonnegative and $\phi(v) \le d(n-1)$ for all $v \in \left(\!\!\binom{\mathscr{F}}{d}\!\!\right)$.

So given $0 \le \alpha \le d(n-1)$, we need to construct a multiset $v \in M_d$ with $\phi(v) = \alpha$. Let $\alpha = i(n-1) + j$ with $0 \le j < n-1$. Then clearly $0 \le i \le d$. If $i = d$ then $j = 0$ and we can use the multiset $v = \{n-1,\ldots,n-1\}$ that contains d copies of the symbol $n-1$. If $i < d$ we use the multiset $v = \{n-1,\ldots,n-1,j,0,\ldots,0\}$ that contains i copies of the symbol $n-1$, one copy of the symbol j, and $d-i-1$ copies of the symbol 0. In either case, we have $\phi(v) = \alpha$. □

Theorem 6.2.2. *For all* $1 \le d \le n$*, there exists a complete* (d,n)*-blueprint with diameter* $d(n-1)$*. That is,* $\overline{\Delta_{BP}}(d,n) = d(n-1)$*.*

Proof. Let $\mathscr{V} := \left(\!\left(\begin{matrix} \mathscr{F} \\ d \end{matrix} \right)\!\right)$ be the set of all d-element multisets in $\mathbb{N}^{\mathscr{F}}$. Define

$$\mathscr{L} := \left(\phi^{-1}(\{0\}), \phi^{-1}(\{1\}), \ldots, \phi^{-1}(\{d(n-1)\})\right)$$

Lemma 6.2.1 says that this is a partition of $\mathscr{V}$, so $\mathscr{L}$ is certainly complete. Furthermore, it has the right diameter. It remains to show that $\mathscr{L}$ is a blueprint. By proposition 3.3.4, it is enough to show that every multiset u of symbols is alive in a contiguous range of levels.

There is nothing to show for multisets u with $|u| \geq d$, so let u be a multiset containing fewer than d symbols. Intuitively, we now fix $|u|$ chips at positions corresponding to u and then perform all possible moves involving only the remaining chips.

Let $k = d - |u|$. Let $v \supset u$ be a vertex covering u in some level $\phi(v)$. Then $\phi(v) = \phi(u) + \phi(x)$, where x is a k-element multiset so that

$$\phi(u) \leq \phi(v) \leq \phi(u) + k(n-1)$$

by lemma 6.2.1. Conversely, let i be an integer such that $\phi(u) \leq \phi(u) + i \leq \phi(u) + k(n-1)$. Then there is a k-element multiset x with $\phi(x) = i$ by lemma 6.2.1. With $v = u \cup x$ we have $\phi(v) = \phi(u) + \phi(x) = \phi(u) + i$. That is, the vertex v lies in level $\phi(u) + i$ and therefore u is alive in level $\phi(u) + i$.

In conclusion, u is alive in exactly the range $[\phi(u), \phi(u) + k(n-1)]$ of levels. Since u was arbitrary, every multiset of symbols is alive in a contiguous range of levels. □

Corollary 6.2.3. *For every $d \geq 1$ there exist infinitely many n with $\overline{\Delta_{APG}}(d,n) \geq dn - 2d^2\sqrt{n}$.*

Proof. Exactly the same as the proof of theorem 5.4.2, except that one now begins with a *complete* blueprint based on theorem 6.2.2. □

7 Summary

In this thesis, I have given an overview over combinatorial abstractions related to the diameter of polytopes and polyhedra. Two of those abstractions had been developed by Adler and Dantzig [AD74] and Kalai [Kal92], respectively. I have also introduced and studied two new abstractions called abstract polyhedral graphs and blueprints which are essentially two different flavours of a common idea. In an excursion, I have introduced families of disjoint coverings, a new structure related to combinatorial designs. My results are:

- The best known general upper bounds on the diameter of polyhedra are still true in the very general setting of blueprints.
- Abstract polyhedral graphs and blueprints are polynomially equivalent in that the diameter of one is bounded by a polynomial if and only if the diameter of the other is bounded by a polynomial as well.
- The Hirsch bound (in fact every linear bound in n) is false for abstract polyhedral graphs and blueprints.
- The maximum diameter of complete blueprints is known precisely and the maximum diameter of complete abstract polyhedral graph is known asymptotically.
- I have given constructive proofs for the existence of large families of disjoint coverings. In the special case $k = r + 1$, I have given an asymptotically optimal construction.

The Hirsch conjecture for polytopes and the question whether there is a polynomial upper bound on the diameter of convex polyhedra remain open. Similarly, the maximum diameters of general blueprints and abstract polyhedral graphs are unknown. My feeling is that the upper bounds in the complete case are true in the general case as well, that is:

Conjecture 7.0.4. $\Delta_{APG}(d, n) \leq d(n-d)$.

Conjecture 7.0.5. $\Delta_{BP}(d, n) \leq d(n-1)$.

However, my failure to find any counterexamples is the only evidence I have for these conjectures. A number of other problems relating to abstract polyhedral graphs can be imagined, such as an analogue to the d-revisiting path conjecture:

Problem 7.0.6. *Given a blueprint or abstract polyhedral graph $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_\ell)$, is there a path, that is a choice $v_j \in \mathscr{L}_j$ for every level j, such that every symbol occurs at most d times in the differences $v_{j+1} \setminus v_j$?*

Note that the *nonrevisiting* path conjecture is false for abstract polyhedral graphs. An affirmative answer to this problem would immediately result in a polynomial bound for the diameter of polyhedra. Another question that may be easier to answer is:

Problem 7.0.7. *Are ultraconnected families and abstract polyhedral graphs polynomially equivalent? In other words, is there a reduction that constructs an ultraconnected family of sets out of arbitrary abstract polyhedral graphs? If this is not possible in general, is there some simple criterion for when it is possible? A similar question can be asked about abstract polyhedra.*

Bibliography

[AD74] Ilan Adler and George B. Dantzig. Maximum diameter of abstract polytopes. *Mathematical Programming Study, Pivoting and Extensions*, pages 20–40, 1974.

[ADM74] Ilan Adler, George B. Dantzig, and Katta Murty. Existence of a-avoiding paths in abstract polytops. *Mathematical Programming Study, Pivoting and Extensions*, pages 41–42, 1974.

[AKS87] Ilan Adler, Richard M. Karp, and Ron Shamir. A simplex variant solving an m×d linear program in $O(\min(m^2, d^2))$ expected number of pivot steps. *J. Complex.*, 3(4):372–387, 1987.

[And97] Ian Anderson. *Combinatorial Designs and Tournaments*. Oxford University Press, 1997.

[Bor87] Karl Heinz Borgwardt. *The Simplex Method: A Probabilistic Analysis*. Springer-Verlag, 1987.

[BR93] Michel L. Balinski and Fred J. Rispoli. Signature classes of transportation polytopes. *Math. Program.*, 60(2):127–144, 1993.

[CD06] Charles J. Colbourn and Jeffrey H. Dinitz, editors. *Handbook of Combinatorial Designs, Second Edition (Discrete Mathematics and Its Applications)*. Chapman & Hall/CRC, 2006.

[Chv83] Vašek Chvátal. *Linear Programming*. W. H. Freeman and Company, 1983.

[Dan63] George B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, 1963.

[Den74] R.H.F. Denniston. Sylvester's problem of the 15 schoolgirls. *Discrete Mathematics*, 9:229–233, 1974.

[Die00] Reinhard Diestel. *Graph theory*. Springer-Verlag, New York, 2 edition, 2000.

[Doy72] Jean Doyen. Constructions of disjoint steiner triple systems. *Proceedings of the American Mathematical Society*, 32(2):409–416, 1972.

[DS92] Jeffrey H. Dinitz and Douglas R. Stinson, editors. *Contemporary Design Theory: A Collection of Surveys*. John Wiley & Sons, Inc., 1992.

[Gal63] David Gale. Neighborly and cyclic polytopes. In Victor Klee, editor, *Convexity*, volume 7 of *Proc. Symp. Pure Math*, pages 225–232. American Mathematical Society, 1963.

[GGL95] Ronald L. Graham, Martin Grötschel, and László Lovász, editors. *Handbook of Combinatorics*, volume 1. Elsevier Science B.V., 1995.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.

[GPKS96] Daniel M. Gordon, Oren Patashnik, Greg Kuperberg, and Joel H. Spencer. Asymptotically optimal covering designs. *J. Comb. Theory Ser. A*, 75(2):270–280, 1996.

[Grü67] Branko Grünbaum. *Convex Polytopes*. Wiley Interscience Publ., London, 1967.

[Gä98] Bernd Gärtner. Combinatorial linear programming: Geometry can help. In *RANDOM '98: Proceedings of the Second International Workshop on Randomization and Approximation Techniques in Computer Science*, pages 82–96, London, UK, 1998. Springer-Verlag.

[Hal35] Philip Hall. On Representatives of Subsets. *J. London Math. Soc.*, s1-10(37):26–30, 1935.

[Hol03] Fred B. Holt. Maximal nonrevisiting paths in simple polytopes. *Discrete Math.*, 263(1-3):105–128, 2003.

[Iwa05] Satoru Iwata. Edge-coloring (lecture in discrete methods in informatics). http://www.misojiro.t.u-tokyo.ac.jp/~iwata/dmi/dmi02e.pdf (retrieved on February 28, 2008), October 2005.

[Kal91] Gil Kalai. The diameter of graphs of convex polytopes and f-vector theory. In *Applied Geometry and Discrete Mathematics*, volume 4 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 387–411. American Mathematical Society, 1991.

[Kal92] Gil Kalai. Upper bounds for the diameter and height of graphs of convex polyhedra. *Discrete & Computational Geometry*, 8:363–372, 1992.

[KK87] Victor Klee and Peter Kleinschmidt. The d-step conjecture and its relatives. *Math. Oper. Res.*, 12(4):718–755, 1987.

[KK92] Gil Kalai and Daniel J. Kleitman. A quasi-polynomial bound for the diameter of graphs of polyhedra. *BAMS: Bulletin of the American Mathematical Society*, 26, 1992.

[Kle64] Victor Klee. Diameters of polyhedral graphs. *Canad. J. Math.*, 16(3):602–614, 1964.

[Kle66] Victor Klee. Paths on polyhedra. ii. *Pacific J. Math*, 17(2):249–262, 1966.

[KW67] Victor Klee and David W. Walkup. The d-step conjecture for polyhedra of dimension $d < 6$. *Acta Math. 133*, pages 53–78, 1967.

[Lar70] David G. Larman. Paths on polytopes. *Proceedings London Math. Soc.*, pages 161–178, 1970.

[Mat94] Jiří Matoušek. Lower bounds for a subexponential optimization algorithm. *Random Structures Algorithms*, 5(4):591–607, 1994.

[MG07] Jiří Matoušek and Bernd Gärtner. *Understanding and Using Linear Programming*. Springer, Berlin, 2007.

[Nad89] Denis Naddef. The Hirsch conjecture is true for $(0,1)$-polytopes. *Math. Programming*, 45:109–110, 1989.

[Rö85] Vojtěch Rödl. On a packing and covering problem. *European Journal of Combinatorics*, (5):69–78, 1985.

[ST04] Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *J. ACM*, 51(3):385–463, 2004.

[SW92] Micha Sharir and Emo Welzl. A combinatorial bound for linear programming and related problems. In *STACS '92: Proceedings of the 9th Annual Symposium on Theoretical Aspects of Computer Science*, pages 569–579, London, UK, 1992. Springer-Verlag.

[Tei73] Luc Teirlinck. On the maximum number of disjoint steiner triple systems. *Discrete Mathematics*, 6:299–300, 1973.

[Ver06] Roman Vershynin. Beyond hirsch conjecture: Walks on random polytopes and smoothed complexity of the simplex method. In *FOCS '06: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science*, pages 133–142, Washington, DC, USA, 2006. IEEE Computer Society.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.

Ehrenwörtliche Erklärung

Hiermit versichere ich, die vorliegende Arbeit ohne Hilfe Dritter und nur mit den angegebenen Quellen und Hilfsmitteln angefertigt zu haben. Alle Stellen, die aus den Quellen entnommen wurden, sind als solche kenntlich gemacht worden. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Paderborn, den 2. Oktober 2008

Nicolai Hähnle