Mock Exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ	
max points:	10	10	10	10	10	10	10	60	
achieved points:									
chosen exercises:									

Remark:
The exercises in this practice exam are intended to give you an idea about how the final exam will look like. Note that these exercises cover a subset of the topics seen in class. We advise to refresh your memory on all the topics covered. The solutions will be published in two weeks. In the final exam, you will be able to choose 6 out of 7 exercises to work on.

Regulations for the final exam:

- Duration: 3 hours.
- Check whether the exam is complete: It should have 8 pages (Exercises 1–7).
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1: (Algorithms)
Given $p \in \mathbb{R}^n_+$, $w \in \mathbb{R}^n_+$ and $b \in \mathbb{R}_+$, design an algorithm which, in $O(n \log n)$ operations, computes the optimal solution x^* to the following linear program:

$$\begin{aligned} \max \quad & p^T x \\ \text{s.t.} \quad & w^T x \leq b, \\ & 0 \leq x_i \leq 1, \quad \forall i \in [n]. \end{aligned}$$

You may assume that a set of n real numbers can be sorted in time $O(n \log n)$ and that each arithmetic operation takes constant time.

Solution:

Use the next page if you need more space

Exercise 2 (Polyhedral theory)

Let

$$\max\{c^T x \ : \ Ax \leq b\} \tag{1}$$

be a feasible linear program. Show that (1) is bounded if and only if the program

$$\max\{c^T x \ : \ Ax \leq \mathbf{0},\ c^T x \leq 1\} \tag{2}$$

has optimal value equal to 0.

Solution:

Use the next page if you need more space

Exercise 3 (Farkas' lemma)

Prove the so called affine form of Farkas' lemma. Suppose that the system $Ax \leq b$ is feasible and that each feasible solution x satisfies $c^T x \leq \delta$. Then there exists a vector $\lambda \geq 0$ such that $\lambda^T A = c$ and $\lambda^T b \leq \delta$.

Solution:

Use the next page if you need more space

Exercise 4 (Simplex phase II)

Consider the following LP:

$$
\begin{array}{lrcrcrcl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
 & -y_1 & + & 4y_2 & + & 2y_3 & \leq & 5 \\
 & 2y_1 & - & 6y_2 & - & y_3 & \leq & 2 \\
 & 2y_1 & - & 3y_2 & + & 4y_3 & \leq & 1 \\
 & -y_1 & & & & & \leq & 0 \\
 & -y_2 & & & & & \leq & 0 \\
 & -y_3 & & & & & \leq & 0
\end{array}
$$

Solve the linear program using the simplex algorithm with smallest index rule (i.e., at each iteration choose i^* to be the smallest index i such that $\lambda_i < 0$).
Start with the basis $B = \{4,5,6\}$ and the corresponding vertex $(0,0,0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

At the end provide the optimal vertex, its objective function value and the certificate of optimality.

The inverse matrices of all feasible bases are:

$$B = \{1,3,4\} \implies A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/11 & -1/11 & -4/11 \\ 3/22 & 2/11 & 5/22 \end{bmatrix} \quad B = \{1,3,6\} \implies A_B^{-1} = \begin{bmatrix} 3/5 & 4/5 & 22/5 \\ 2/5 & 1/5 & 8/5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \{1,4,6\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/4 & -1/4 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{3,4,5\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}$$

$$B = \{3,5,6\} \implies A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{4,5,6\} \implies A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution:

Use the next page if you need more space

Exercise 5 (Complementary slackness/Duality)

(a) Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, let **P** be a linear program of the form $\max\{c^\top x : Ax \leq b\}$ and **D** be its dual $\min\{b^\top y : A^\top y = c, y \geq 0\}$. Furthermore, assume that **P** is feasible and bounded, and x^*, y^* are optimal solutions of **P**, **D** respectively. Prove that the following statement holds for every $j \in [m]$:

If $y_j^* > 0$, then x^* is active at the j-th constraint, i.e. $a_j x^* = b_j$.

(b) Consider the following linear program:

$$\begin{aligned} \min \quad & x_1 + 2x_2 + x_3 - x_4 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 - x_4 = 1 \\ & 2x_1 - x_2 + x_4 = 2 \\ & x_3 - x_4 = -1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \\ & x_4 \geq 0 \end{aligned}$$

By using part a) and duality, show that $x^* = (1, 1, 0, 1)$ is an optimal solution.

Solution:

Use the next page if you need more space

Exercise 6 (Ellipsoid)

Consider an affine map $t(x) = Bx + b$ and an ellipsoid

$$\mathscr{E}(A,a) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \le 1\}, \tag{3}$$

where $A, B \in \mathbb{R}^{n \times n}$ are symmetric positive definite (PD) matrices, and $a, b \in \mathbb{R}^n$. Provide a description of the ellipsoid $\mathscr{E}(\bar{A}, \bar{a})$ which is the image of $\mathscr{E}(A,a)$ under $t(x)$.

Solution:

Use the next page if you need more space

Exercise 7 (Separation/Paths)

Recall the separation problem for a polyhedron $P \subset \mathbb{R}^n$: given $\bar{x} \in \mathbb{R}^n$, we want to determine whether $\bar{x} \in P$ and if not, to compute an inequality $c^T x \leq d$ which is valid for P, and which for $c^T \bar{x} > d$.

(a) Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, with $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$. Show that the separation problem for P can be solved in time polynomial in m, n and the encoding length of A, b (and the encoding length of the input point $\bar{x}$).

(b) Let $D(V,A)$ be a directed graph, and let

$$Q = \{x \in \mathbb{R}^A : \sum_{a \in C} x_a \geq 0, \text{ for all directed cycles } C \text{ in } D(V,A)\}.$$

Let $|V| = n, |A| = m$. In the definition of Q, there is a variable for each arc in A and an inequality for each cycle in $D(V,A)$, and the number of cycles in a graph can be exponential in n, m. Show that we can still solve the separation problem for C in time polynomial in n, m (and the encoding length of the input point $\bar{x}$).

Solution:

Use the next page if you need more space