DISCRETE OPTIMIZATION
PROF. FRIEDRICH EISENBRAND

Spring 2017
14.04.2017

Midterm Exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ
max points:	10	10	10	10	10	10	10	60
achieved points:								
chosen exercises:								

Remark:
The exercises in this practice exam are intended to give you an idea about how the final exam will look like. Note that these exercises cover a subset of the topics seen in class. We advise to refresh your memory on all the topics covered. The solutions will be published after the Easter break. In the final exam, you will be able to choose 6 out of 7 exercises to work on.

Regulations for the final exam:

- Duration: 3 hours.
- Check whether the exam is complete: It should have 8 pages (Exercises 1–7).
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1 (Algorithm design and analysis)

Consider the following problem: we are given $a_1, \dots, a_n \in \mathbb{Q}$, and we want to find i, j such that $1 \le i \le j \le n$ and $\sum_{k=i}^{j} a_k$ is maximized.

(a) Give an algorithm that solves this problem and performs a number of arithmetic and elementary operations that is polynomial in n.

(b) Give an algorithm that solves this problem with $O(n)$ arithmetic and elementary operations, and prove its correctness.

Solution:

Use the next page if you need more space

Exercise 2 (Convexity) The *Minkowski sum* A+B for two sets $A, B \subseteq \mathbb{R}^d$ is defined as $A+B := \{a+b : \ a \in A, \ b \in B\}$. Moreover, we define $2A := \{2a : \ a \in A\}$. Let $A \subseteq \mathbb{R}^d$ be a closed set.

a) Show that A is convex if and only if $A+A = 2A$.

b) For every $d \in \mathbb{N}$ give an example of a closed set $B \subseteq \mathbb{R}^d$ for which $B+B \neq 2B$

Solution:

Use the next page if you need more space

Exercise 3: (Vertices of a polyhedron)

Let $P = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1, i = 1, \ldots, n\}$. Find all the vertices of P (and prove that there are no others).

Solution:

Use the next page if you need more space

Exercise 4 (Polyhedral theory)

Let

$$\max\{c^T x \,:\, Ax \leq b\} \tag{1}$$

be a feasible linear program. Show that (1) is bounded if and only if the program

$$\max\{c^T x \,:\, Ax \leq \mathbf{0},\ c^T x \leq 1\} \tag{2}$$

has optimal value equal to 0.

Solution:

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Exercise 5 (Farkas' lemma)

Prove the so called affine form of Farkas' lemma. Suppose that the system $Ax \leq b$ is feasible and that each feasible solution x satisfies $c^T x \leq \delta$. Then there exists a vector $\lambda \geq 0$ such that $\lambda^T A = c$ and $\lambda^T b \leq \delta$.

Solution:

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Exercise 6 (Complementary slackness)

(a) Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Let **P** be a linear program of the form $\max\{c^\top x : Ax \le b\}$ and let **D** be its dual $\min\{b^\top y : A^\top y = c, y \ge 0\}$, and assume that **P** is feasible and bounded. Let x^*, y^* be optimal solutions for **P**, **D** respectively. Prove that for any $j = 1, \ldots, m$ if $y_j^* > 0$, then x^* is tight at the j-th constraint, i.e. $a_j x^* = b_j$.

(b) Consider the following linear program:

$$\begin{aligned}
\min \quad & x_1 + 2x_2 + x_3 - x_4 \\
\text{s.t.} \quad & x_1 + x_2 - x_3 - x_4 = 1 \\
& 2x_1 - x_2 + x_4 = 2 \\
& x_3 - x_4 = -1 \\
& x_1 \ge 0 \\
& x_2 \ge 0 \\
& x_3 \ge 0 \\
& x_4 \ge 0
\end{aligned}$$

By using part 1) and strong duality, show that $x^* = (1, 1, 0, 1)$ is an optimal solution.

Solution:

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Exercise 7 (Simplex phase II)

Consider the following LP:

$$
\begin{array}{lrcrcrcl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
 & -y_1 & + & 4y_2 & + & 2y_3 & \leq & 5 \\
 & 2y_1 & - & 6y_2 & - & y_3 & \leq & 2 \\
 & 2y_1 & - & 3y_2 & + & 4y_3 & \leq & 1 \\
 & -y_1 & & & & & \leq & 0 \\
 & -y_2 & & & & & \leq & 0 \\
 & -y_3 & & & & & \leq & 0
\end{array}
$$

Solve the linear program using the simplex algorithm with smallest index rule (i.e., at each iteration choose i^* to be the smallest index i such that $\lambda_i < 0$).
Start with the basis $B = \{4,5,6\}$ and the corresponding vertex $(0,0,0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

At the end provide the optimal vertex, its objective function value and the certificate of optimality.

The inverse matrices of all feasible bases are:

$$B = \{1,3,4\} \implies A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/11 & -1/11 & -4/11 \\ 3/22 & 2/11 & 5/22 \end{bmatrix} \qquad B = \{1,3,6\} \implies A_B^{-1} = \begin{bmatrix} 3/5 & 4/5 & 22/5 \\ 2/5 & 1/5 & 8/5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \{1,4,6\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/4 & -1/4 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \{3,4,5\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}$$

$$B = \{3,5,6\} \implies A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \{4,5,6\} \implies A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution:

Use the next page if you need more space