DISCRETE OPTIMIZATION
PROF. FRIEDRICH EISENBRAND

Spring 2018
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Mock Exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ	
max points:	10	10	10	10	10	10	10	60	
achieved points:									
chosen exercises:									

Remark:
The exercises in this practice exam are intended to give you an idea about how the final exam will look like. Note that these exercises cover a subset of the topics seen in class. We advise to refresh your memory on all the topics covered. The solutions will be published in two weeks. In the final exam, you will be able to choose 6 out of 7 exercises to work on.

Regulations for the final exam:

- Duration: 3 hours.
- Check whether the exam is complete: It should have 9 pages (Exercises 1–7).
- Write your name on the title page. Put your CAMIPRO card on your table.
- Use neither pencil nor red colored pen!
- Solutions have to be written below the exercises. Solutions must be comprehensible.
- In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

No additional aids are allowed to the exam

Exercise 1: (Algorithms)
Given $p \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^n$ and $b \in \mathbb{R}_+$, design an algorithm which, in $O(n\log n)$ operations, computes the optimal solution x^* to the following linear program:

$$\begin{aligned}
\max \quad & p^T x \\
\text{s.t.} \quad & w^T x \le b, \\
& 0 \le x_i \le 1, \quad \forall i \in [n].
\end{aligned}$$

You may assume that a set of n real numbers can be sorted in time $O(n\log n)$ and that each arithmetic operation takes constant time.

Solution:
We assume that the bove LP is feasible, which is easy to check. Now, sort items according to their profit/weight ratio in time $O(n\log n)$ and re-index them so that

$$\frac{p_1}{w_1} \ge \frac{p_2}{w_2} \ge \cdots \ge \frac{p_n}{w_n}.$$

Let k be the maximum index in $[n]$ so that

$$\sum_{i=1}^{k} w_i \le b.$$

Define a solution $\bar{x}$ with $\bar{x}_i = 1$ for $i \le k$, $\bar{x}_{k+1} = \frac{b-\sum_{i=1}^k w_i}{w_{k+1}}$ and $\bar{x}_i = 0$ for $i > k+1$. We will prove that $\bar{x}$ is an optimal solution to the above LP.
Let x^* be an arbitrary feasible solution, we would like to show that $p^T x^* \le p^T \bar{x}$, i.e.,

$$\sum_{i>k} p_i x_i^* \le \sum_{i\le k} p_i(\bar{x}_i - x_i^*) + \sum_{i>k} p_i \bar{x}_i.$$

We have that:

$$\sum_{i>k} p_i x_i^* = \sum_{i>k} \frac{p_i}{w_i} w_i x_i^* \le \sum_{i>k} \frac{p_{k+1}}{w_{k+1}} w_i x_i^* = \frac{p_{k+1}}{w_{k+1}} \sum_{i>k} w_i x_i^*.$$

Similarly, we obtain that:

$$\begin{aligned}
\sum_{i\le k} p_i(\bar{x}_i - x_i^*) + \sum_{i>k} p_i \bar{x}_i &= \sum_{i\le k} p_i (1-x_i^*) + p_{k+1} \frac{b-\sum_{i=1}^k w_i}{w_{k+1}} \\
&= \sum_{i\le k} \frac{p_i}{w_i}(w_i - w_i x_i^*) + \frac{p_{k+1}}{w_{k+1}}(b - \sum_{i=1}^{k} w_i) \\
&\ge \frac{p_{k+1}}{w_{k+1}} \sum_{i\le k}(w_i - w_i x_i^*) + \frac{p_{k+1}}{w_{k+1}}(b - \sum_{i=1}^{k} w_i) \\
&= \frac{p_{k+1}}{w_{k+1}}(b - \sum_{i\le k} w_i x_i^*) \\
&\ge \frac{p_{k+1}}{w_{k+1}} \sum_{i>k} w_i x_i^*,
\end{aligned}$$

where the last inequality follows from the fact that x^* is feasible.

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Exercise 2 (Polyhedral theory)

Let

$$\max\{c^T x \;:\; Ax \le b\} \tag{1}$$

be a feasible linear program. Show that (1) is bounded if and only if the program

$$\max\{c^T x \;:\; Ax \le \mathbf{0},\; c^T x \le 1\} \tag{2}$$

has optimal value equal to 0.

Solution:

Linear program (1) is unbounded if and only if there is a half line contained in $\{Ax \le b\}$, i.e. there are $x,\ y \in \mathbb{R}^n$ such that $Ay \le b$, $A(\lambda x + y) \le b$ for any $\lambda \ge 0$, and $c^T x > 0$. This happens if and only if there is an x such that $Ax \le 0$ and $c^T x > 0$, which is equivalent to saying that (2) has optimum value greater than 0 (the point x can be scaled such that $c^T x \le 1$).

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Exercise 3 (Farkas' lemma)

Prove the so called affine form of Farkas' lemma. Suppose that the system $Ax \leq b$ is feasible and that each feasible solution x satisfies $c^T x \leq \delta$. Then there exists a vector $\lambda \geq 0$ such that $\lambda^T A = c$ and $\lambda^T b \leq \delta$.

Solution:

Let $\max\{c^T x : Ax \leq b\}$ and $\min\{y^T b : A^T y = c, y \geq 0\}$ be the primal and the dual, respectively. Since $c^T x \leq \delta$ we know that the primal is bounded. Then by strong duality there exist primal feasible x^* and dual feasible y^* such that $c^T x^* = b^T y^*$. Set $\lambda = y^*$, then primal feasibility gives $\lambda^T b = c^T x^* \leq \delta$ and by dual feasibility $\lambda^T A = c, \lambda \geq 0$.

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Exercise 4 (Simplex phase II)

Consider the following LP:

$$\begin{array}{rrcrcrcl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
 & -y_1 & + & 4y_2 & + & 2y_3 & \leq & 5 \\
 & 2y_1 & - & 6y_2 & - & y_3 & \leq & 2 \\
 & 2y_1 & - & 3y_2 & + & 4y_3 & \leq & 1 \\
 & -y_1 & & & & & \leq & 0 \\
 & -y_2 & & & & & \leq & 0 \\
 & -y_3 & & & & & \leq & 0
\end{array}$$

Solve the linear program using the simplex algorithm with smallest index rule (i.e., at each iteration choose i^* to be the smallest index i such that $\lambda_i < 0$).
Start with the basis $B = \{4,5,6\}$ and the corresponding vertex $(0,0,0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

At the end provide the optimal vertex, its objective function value and the certificate of optimality.

The inverse matrices of all feasible bases are:

$$B = \{1,3,4\} \implies A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/11 & -1/11 & -4/11 \\ 3/22 & 2/11 & 5/22 \end{bmatrix} \quad B = \{1,3,6\} \implies A_B^{-1} = \begin{bmatrix} 3/5 & 4/5 & 22/5 \\ 2/5 & 1/5 & 8/5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \{1,4,6\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/4 & -1/4 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{3,4,5\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}$$

$$B = \{3,5,6\} \implies A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{4,5,6\} \implies A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solution:

iter	basis	vertex
0	$\{4,5,6\}$	$(0,0,0)^T$
1	$\{3,5,6\}$	$(1/2,0,0)^T$
2	$\{1,3,6\}$	$(19/5,11/5,0)^T$

The optimum is reached at the vertex $(19/5,11/5,0)^T$, the corresponding objective value is $41/5$ and the certificate is the vector

$$\lambda^T = (7/5,0,6/5,0,0,23/5).$$

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Exercise 5 (Complementary slackness/Duality)

(a) Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, let **P** be a linear program of the form $\max\{c^\top x : Ax \leq b\}$ and **D** be its dual $\min\{b^\top y : A^\top y = c, y \geq 0\}$. Furthermore, assume that **P** is feasible and bounded, and x^*, y^* are optimal solutions of **P**, **D** respectively. Prove that the following statement holds for every $j \in [m]$:

If $y_j^* > 0$, then x^* is active at the j-th constraint, i.e. $a_j x^* = b_j$.

(b) Consider the following linear program:

$$\begin{aligned}
\min \quad & x_1 + 2x_2 + x_3 - x_4 \\
\text{s.t.} \quad & x_1 + x_2 - x_3 - x_4 = 1 \\
& 2x_1 - x_2 + x_4 = 2 \\
& x_3 - x_4 = -1 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& x_4 \geq 0
\end{aligned}$$

By using part a) and duality, show that $x^* = (1,1,0,1)$ is an optimal solution.

Solution:

(a) By strong duality we have that $c^\top x^* = b^\top y^*$. By exploiting that any scalar coincides with its transpose and plugging in $A^\top y^* = c$ we get:

$$c^\top x^* = x^{*\top} c = x^{*\top} A^\top y^* = b^\top y^*,$$

which implies

$$(b - Ax^*)^\top y^* = 0.$$

Since $b - Ax^*$, y^* are non-negative, we must have that for any j, either $y_j^* = 0$, or $b_j - a_j x^* = 0$, which proves the thesis.

(b) First, it is immediate to verify that x^* is feasible. We then compute the dual program:

$$\begin{aligned}
\max \quad & y_1 + 2y_2 - y_3 \\
\text{s.t.} \quad & y_1 + 2y_2 \leq 1 \\
& y_1 - y_2 \leq 2 \\
& -y_1 + y_3 \leq 1 \\
& -y_1 + y_2 - y_3 \leq -1
\end{aligned}$$

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Solution:

Now, by part 1) (and exchanging primal and dual) we know that, if x^* is an optimal solution, any optimal solution of the dual y^* must satisfy with equality the first, the second and the fourth constraint. By turning these three constraints into equalities we get a system with a unique solution $y^* = (\frac{5}{3}, \frac{-1}{3}, -1)$. Now we verify that y^* is feasible and that yields an objective value of 2, the same as x^*. By strong duality, this implies that x^* is an optimal solution.

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Exercise 6 (Ellipsoid)

Consider an affine map $t(x) = Bx + b$ and an ellipsoid

$$\mathscr{E}(A,a) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \leq 1\}, \tag{3}$$

where $A, B \in \mathbb{R}^{n\times n}$ are symmetric positive definite (PD) matrices, and $a, b \in \mathbb{R}^n$. Provide a description of the ellipsoid $\mathscr{E}(\bar{A},\bar{a})$ which is the image of $\mathscr{E}(A,a)$ under $t(x)$.

Solution:
The desired ellipsoid is:

$$\begin{aligned}
\{t(x) \in \mathbb{R}^n \mid x \in \mathscr{E}(A,a)\} &= \{\underbrace{t(x)}_{y} \in \mathbb{R}^n \mid (\underbrace{x}_{t^{-1}(y)} - a)^T A^{-1}(\underbrace{x}_{t^{-1}(y)} - a) \leq 1\} && (4)\\
&= \{y \in \mathbb{R}^n \mid (B^{-1}(y-b)-a)^T A^{-1}(B^{-1}(y-b)-a) \leq 1\} && (5)\\
&= \{y \in \mathbb{R}^n \mid (y-b-Ba)^T B^{-T} A^{-1} B^{-1}(y-b-Ba) \leq 1\} && (6)\\
&= \{y \in \mathbb{R}^n \mid (y-(Ba+b))^T (BAB)^{-1}(y-(Ba+b)) \leq 1\}. && (7)
\end{aligned}$$

Thus, the ellipsoid $\mathscr{E}(\bar{A},\bar{a})$ is defined by $\bar{A} = BAB$ and $\bar{a} = Ba + b$.

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Exercise 7 (Separation/Paths)

Recall the separation problem for a polyhedron $P \subset \mathbb{R}^n$: given $\bar{x} \in \mathbb{R}^n$, we want to determine whether $\bar{x} \in P$ and if not, to compute an inequality $c^T x \leq d$ which is valid for P, and which for $c^T \bar{x} > d$.

(a) Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, with $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$. Show that the separation problem for P can be solved in time polynomial in m, n and the encoding length of A, b (and the encoding length of the input point $\bar{x}$).

(b) Let $D(V,A)$ be a directed graph, and let

$$Q = \{x \in \mathbb{R}^A : \sum_{a \in C} x_a \geq 0, \text{ for all directed cycles } C \text{ in } D(V,A)\}.$$

Let $|V| = n, |A| = m$. In the definition of Q, there is a variable for each arc in A and an inequality for each cycle in $D(V,A)$, and the number of cycles in a graph can be exponential in n, m. Show that we can still solve the separation problem for C in time polynomial in n, m (and the encoding length of the input point $\bar{x}$).

Solution:

a) To know whether a given input point $\bar{x}$ is in P, we can just verify that $A\bar{x} \leq b$. This amounts to computing a matrix - vector product, which can be done with mn operations, plus m comparisons, hence it can be done in polynomial time in m, n and the size of the input. If $A_j \bar{x} > b_j$ for some $j \in [m]$, then $A_j x \leq b_j$ is a separating inequality.

b) Notice that we cannot use part a) directly since the number of inequalities describing Q is exponential in m, n. However, given $\bar{x} \in \mathbb{R}^A$, we can see $\bar{x}$ as a length function on the arcs of D and detect whether there is any negative cycle with respect to $\bar{x}$ using the Bellman-Ford algorithm. To be precise, we add a vertex s and an arc (s, v) for every $v \in V$, and then run Bellman-Ford on this graph. The algorithm (which runs in polynomial time since our graph has size polynomial in the size of D) will either find a directed cycle C such that $\sum_{a \in C} \bar{x}_a < 0$, in which case $\bar{x} \notin Q$, or certify that none exists, i.e. $\bar{x} \in Q$.

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