Part 2
Mean-Variance Portfolio Theory

Further reading

INVESTMENT SCIENCE, by David G. Luenberger, Oxford University Press

Today chapter 6

Assets

Asset is investment instrument that can be bought and sold frequently

Return

- X_0: amount invested, X_1: amount received
- Total return: $R = \frac{X_1}{X_0}$
- Rate of return: $r = \frac{X_1 - X_0}{X_0}$

Short sales

Sell asset without owning it.

- Borrow asset from someone who owns it (such as a brokerage firm)
- sell borrowed asset to someone else receiving X_0
- Repay your loan by purchasing asset for X_1
- Return asset to lender
- Short selling is profitable if the asset price declines

Portfolios

Portfolio return

- n assets are available with return R_i and X_0 is to be invested
- weight w_i: fraction of asset i in portfolio
- $\sum_{i=1}^{n} w_i = 1$
- Return of Portfolio:

$$
\begin{aligned}
R &= \frac{\sum_{i=1}^{n} w_i X_0 R_i}{X_0} \\
&= \sum_{i=1}^{n} w_i R_i
\end{aligned}
$$

- Using formula $R = 1 + r$ and $\sum_{i=1}^{n} w_i = 1$, one has $r = \sum_{i=1}^{n} w_i r_i$ for the rate of return

Basic notions of probability

Expected value, variance

- x a random variable over a finite probability space, expected value $E(x)$ or $\overline{x}$ of x is $\sum_i x_i p_i$, where p_i is the probability of x to attain value x_i
- Linearity of expectation: x, y random variables, $\alpha, \beta \in \mathbb{R}$, then $E(\alpha x + \beta y) = \alpha E(x) + \beta E(y)$
- Variance of x: $\mathrm{var}(x) = E[(x - \overline{x})^2] = E(x^2) - \bar{x}^2$
- Standard deviation: $\sigma(x) = \sqrt{\mathrm{var}(x)}$

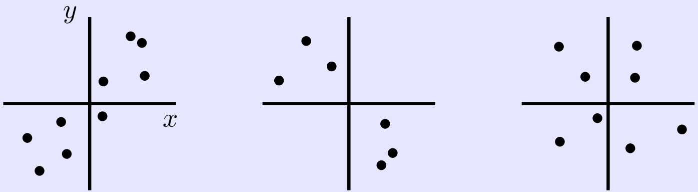
Example: Rolling a dice

$$\begin{aligned}
\sigma^2(x) &= E(x^2) - 3.5^2 \\
&= 1/6(1+4+9+16+25+36) - 3.5^2 \\
&= 2.92
\end{aligned}$$

Covariance

- Covariance: $\mathrm{cov}(x,y) = E[(x-\overline{x})(y-\overline{y})] = E(x\cdot y) - \overline{x}\cdot\overline{y}$
- Correlation: $\rho(x,y) = \mathrm{cov}(x,y)/(\sigma(x)\cdot\sigma(y))$; notice $|\rho(x,y)| \leqslant 1$
- Uncorrelated: $\rho(x,y) = 0$
- Positively correlated: $\rho(x,y) > 0$
- Negatively correlated: $\rho(x,y) < 0$

Example of samples of positive, negative and uncorrelated variables



Variance of sum

$$
\begin{aligned}
\operatorname{var}(\sum_{i=1}^{n} x_i) &= E[\big(\sum_{i=1}^{n} x_i - \overline{x}_i\big)^2] \\
&= E[\sum_{i,j} (x_i - \overline{x}_i)(x_j - \overline{x}_j)] \\
&= \sum_{i,j=1}^{n} E(x_i - \overline{x}_i)(x_j - \overline{x}_j)) \\
&= \sum_{i,j=1}^{n} \operatorname{cov}(x_i, y_i).
\end{aligned}
$$

Random returns

- Rates of return are considered to be random variables
- n assets with random rates of return r_i $i = 1, \ldots, n$ and expected value $\overline{r}_i$ respectively
- Expected rate of return of portfolio

$$\overline{r} = \sum_{i=1}^{n} \overline{r}_i w_i$$

- Variance of portfolio return

$$\text{var}(r) = \sum_{i,j=1}^{n} w_i w_j \text{var}(r_i, r_j) = w^T Q w$$

where Q is symmetric positive semi-definite matrix

The optimization problem

Task

Compute weights such that variance (risk) of portfolio is minimal among those having at least a certain expected return.

Quadratic programming

Convex quadratic program

$Q \in \mathbb{R}^{n \times n}$ positive semidefinite, $c \in \mathbb{R}^n$ $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then

$$\begin{aligned} \min x^T Q x + c^T x \\ Ax &= b \\ x &\geqslant 0, \end{aligned}$$

is (convex) quadratic program.

Efficient algorithms

Convex quadratic programs can be solved efficiently.

Mean-variance portfolio optimization as a quadratic program

$$\min w^T Q w$$

$$\begin{aligned} \sum_{i=1}^{n} w_i \bar{r}_i &\geqslant R \\ \sum_{i=1}^{n} w_i &= 1 \\ w &\geqslant 0 \text{ if no shortsales allowed} \\ Aw &\leqslant b \text{ additional constraints} \end{aligned}$$

Parameter estimation

Observed rates of return

- Time series: $r_1, \ldots, r_T$
- Mean value: $\overline{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$
- Variance: $\mathrm{var}(r) = \frac{1}{T}\sum_{t=1}^{T}(r_t - \overline{r})^2$
- Second time series: $s_1, \ldots, s_T$, mean: $\overline{s}$, variance $\mathrm{var}(s)$
- Covariance: $\mathrm{cov}(r,s) = \frac{1}{T}\sum_{t=1}^{T}(r_t - \overline{r})(s_t - \overline{s})$
- Correlation: $\rho(r,s) = \mathrm{cov}(r,s)/(\sigma(r)\sigma(s))$

Example

- S&P 500 index for returns on stocks, 10 year treasury bond index for returns on bonds, cash invested in money market a 1 day federal fund rate

Year	Stocks	Bonds	MM
1960	20.2553	262.935	100.00
1961	25.6860	268.730	102.33
1962	23.4297	284.090	105.33
1963	28.7463	289.162	108.89
1964	33.4484	299.894	113.08
1965	37.5813	302.695	117.97
1966	33.7839	318.197	124.34
1967	41.8725	309.103	129.94
1968	46.4795	316.051	137.77
1969	42.5448	298.249	150.12
1970	44.2212	354.671	157.48
1971	50.5451	394.532	164.00
1972	60.1461	403.942	172.74
1973	51.3114	417.252	189.93
1974	37.7306	433.927	206.13
1975	51.7772	457.885	216.85
1976	64.1659	529.141	226.93
1977	59.5739	531.144	241.82
1978	63.4884	524.435	266.07
1979	75.3032	531.040	302.74
1980	99.7795	517.860	359.96
1981	94.8671	538.769	404.48

Year	Stocks	Bonds	MM
1982	115.308	777.332	440.68
1983	141.316	787.357	482.42
1984	150.181	907.712	522.84
1985	197.829	1200.63	566.08
1986	234.755	1469.45	605.20
1987	247.080	1424.91	646.17
1988	288.116	1522.40	702.77
1989	379.409	1804.63	762.16
1990	367.636	1944.25	817.87
1991	479.633	2320.64	854.10
1992	516.178	2490.97	879.04
1993	568.202	2816.40	905.06
1994	575.705	2610.12	954.39
1995	792.042	3287.27	1007.84
1996	973.897	3291.58	1061.15
1997	1298.82	3687.33	1119.51
1998	1670.01	4220.24	1171.91
1999	2021.40	3903.32	1234.02
2000	1837.36	4575.33	1313.00
2001	1618.98	4827.26	1336.89
2002	1261.18	5558.40	1353.47
2003	1622.94	5588.19	1366.73

Annual rates of return

Year	Stocks	Bonds	MM
1961	26.81	2.20	2.33
1962	-8.78	5.72	2.93
1963	22.69	1.79	3.38
1964	16.36	3.71	3.85
1965	12.36	0.93	4.32
1966	-10.10	5.12	5.40
1967	23.94	-2.86	4.51
1968	11.00	2.25	6.02
1969	-8.47	-5.63	8.97
1970	3.94	18.92	4.90
1971	14.30	11.24	4.14
1972	18.99	2.39	5.33
1973	-14.69	3.29	9.95
1974	-26.47	4.00	8.53
1975	37.23	5.52	5.20
1976	23.93	15.56	4.65
1977	-7.16	0.38	6.56
1978	6.57	-1.26	10.03
1979	18.61	-1.26	13.78
1980	32.50	-2.48	18.90
1981	-4.92	4.04	12.37
1982	21.55	44.28	8.95

Year	Stocks	Bonds	MM
1983	22.56	1.29	9.47
1984	6.27	15.29	8.38
1985	31.17	32.27	8.27
1986	18.67	22.39	6.91
1987	5.25	-3.03	6.77
1988	16.61	6.84	8.76
1989	31.69	18.54	8.45
1990	-3.10	7.74	7.31
1991	30.46	19.36	4.43
1992	7.62	7.34	2.92
1993	10.08	13.06	2.96
1994	1.32	-7.32	5.45
1995	37.58	25.94	5.60
1996	22.96	0.13	5.29
1997	33.36	12.02	5.50
1998	28.58	14.45	4.68
1999	21.04	-7.51	5.30
2000	-9.10	17.22	6.40
2001	-11.89	5.51	1.82
2002	22.10	15.15	1.24
2003	28.68	0.54	0.98

Rates of return

$r_{it} = (I_{i,t} - I_{i,t-1}) / I_{i,t-1}$ rates of return of asset $i = 1,,2,3$

Arithmetic mean

$\overline{r}_i = 1/T \cdot \sum_{t=1}^{T} r_{it}$ gives

	Stocks	Bonds	MM
Arithmetic mean $\overline{r}_i$	12.06 %	7.85%	6.32 %

Geometric mean

Suppose I invest X_0 at time 0. Rates of return are $r_1 = .25$ and $r_2 = -.25$ on the first and second day. What is X_2?

$X_2 = (1-.25)(1+.25)X_0 = (1-.0625)X_0$

Arithmetic mean of return is 0.

Geometric mean: $(1+\overline{r})^2 = (1-.25)(1+.25)$ and thus $\overline{r} = \sqrt{(1-.25)(1+.25)} - 1$.

Geometric mean

$\overline{r}_i = \sqrt[T]{\prod_{i=1}^{T}(1+r_{it})} - 1.$

Is constant yearly rate of return to be applied to obtain I_{iT} starting from I_{i0}.

Example with geometric mean

	Stocks	Bonds	MM
Geometric mean $\overline{r}_i$	10.73%	7.37%	6.27%

Covariance matrix

$\text{cov}(R_i, R_j) = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \overline{r}_{it})(r_{jt} - \overline{r}_{jt})$:

Covariance	Stocks	Bonds	MM
Stocks	0.02778	0.00387	0.00021
Bonds	0.00387	0.01112	-0.00020
MM	0.00021	-0.00020	0.00115

Example cont.

Volatility (standard deviation)

$\sigma(R_i) = \sqrt{\text{cov}(R_i, R_i)}$

Volatility	16.67 %	10.55 %	3.40 %

Setting up the quadratic program

$$\min .02778 \cdot w_S^2 + 2 \cdot .00387 w_S w_B + 2 \cdot .00021 w_S w_M$$
$$+ .01112 w_B^2 - 2 \cdot .00020 w_B w_M + 0.00115 w_M^2$$

such that

$$.1073 w_S + .0737 w_B + .0627 w_M \geqslant R$$
$$w_S + w_B + w_M = 1$$
$$w_s \geqslant 0,\ w_B \geqslant 0,\ w_M \geqslant 0$$

Diversification

Markowitz portfolio tends to have unreasonably large quantities of individual assets and if short-sales are allowed, unusually large quantities of short sales.
Example: Limit certain assets to m% means additional constraints

$$w_i \leqslant m, \text{ for assets } i.$$

Limiting the overall amount of short sales:
Split each w_i into $w_i = w_i^+ - w_i-$

$$\sum_{i=1}^{n} w_i^- \leqslant m',$$

These go in additional constraint section of Markowitz model

Transaction costs

Re-optimizing a portfolio

w^0 current portfolio and w the desired portfolio. Let y_i be the fraction of asset i sold and z_i be the fraction of asset i bought.

$$\begin{aligned} w_i - w_i^0 \leqslant y_i, &\quad y_i \geqslant 0 \\ w_i^0 - w_i \leqslant z_i, &\quad z_i \geqslant 0 \end{aligned}$$

Following constraint limits fraction of transaction to h:

$$\sum_{i=1}^{n} (y_i + z_i) \leqslant h$$

Robustness

Markowitz model is sensitive

Small change of mean and variance in input results often in large changes of portfolio.
One way to deal with this is to re-sample returns from historical data to generate alternative, but similar $\overline{r}$ and var estimates. Run optimizer on the generated inputs and take mean.

Project

Due November 4, groups of 5 people

Case study at page 167 in the textbook (Optimization Methods in Finance)
Implement the re-sampling idea to create a robust portfolio.
Advertise for your portfolio with a short talk (10 min., exercise session November 4) in which you explain the methods you used and why your portfolio behaves well in practice.