Part 6

The fundamental Theorem of Asset Pricing

Complementary slackness

Recall LP Duality

Let $\min\{c^T x \mid Ax = b, x \geqslant 0\}$ be an LP in standard form. If LP is feasible and bounded, then also the dual LP $\max\{b^T y \mid A^T y \leqslant c\}$ is feasible and bounded and there exist optimal solutions x^* and y^* of the primal resp. dual with $c^T x^* = b^T y^*$

Complementary Slackness

Let x^* and y^* be feasible solutions of the primal and dual linear program. The following conditions are equivalent:

1. x^* and y^* are optimal solutions of primal and dual respectively.
2. $x^*(i) > 0 \implies (c - A^T y^*)(i) = 0$

Strict complementary slackness

If both primal and dual are feasible, then there exist optimal solutions x^* and y^* of the primal and dual respectively with

$$x^* + (c - A^T y^*) > 0.$$

Arbitrage

Arbitrage is trading strategy that:

1. has positive initial cash flow and no risk of loss later (Type A)
2. requires no initial cash input, has no risk of loss and has positive probability of making profits in the future (Type B)

Generalization of binomial setting

- Let $\omega_1, \dots, \omega_m$ be finite set of possible states
- For securities S^i, $i = 0, \dots, n$ let $S^i_1(\omega_j)$ be price of security in state ω_j at time 1 and let S^i_o be price of security i at time 0
- S^0 is riskless security that pays interest rate $r \geqslant 0$ between time 0 and time 1; $S^0_0 = 1$ and $S^0_1(\omega_j) = R = (1 + r)$ for $j = 1, \dots, m$

Risk-neutral probability

A risk-neutral probability measure on the set $\Omega = \{\omega_1, \dots, \omega_m\}$ is a vector $p_1, \dots, p_m$ of positive numbers with $\sum_{j=1}^m p_j = 1$ and for every $S^i, i = 0, \dots, n$ one has

$$S^i_0 = \frac{1}{R} \sum_{j=1}^{m} p_j S^i_1(\omega_j) = \frac{1}{R} E[S^i_1].$$

Here $E[S^i_1]$ is the expected value of random variable S under probability distribution $(p_1, \dots, p_m)$.

Fundamental theorem of asset pricing

First fundamental theorem of asset pricing

A risk-neutral probability measure exists if and only if there is no arbitrage

PART 6.1
ARBITRAGE DETECTION USING LINEAR PROGRAMMING

Scenario

- Portfolio of derivate securities (European call options) S^i, $i = 1, \ldots, n$ of one security S is determined by non-negative vector $(x_1, \ldots, x_n)$
- Payoff of portfolio is $\Psi^x(S_1) = \sum_{i=1}^n \Psi_i(S_1) x_i$, where $\Psi_i(S_1) = \max\{(S_1 - K_i), 0\}$, where K_i is strike price K_i (piecewise linear function with one breakpoint!)
- Cost of performing portfolio at time 0:

$$\sum_{i=1}^n S_0^i x_i.$$

Determine arbitrage possibility

- Negative cost of portfolio with nonnegative payoff (type A)
- Cost zero and positive payoff (type B)

Observation

Nonnegative payoff

Payoff is piecewise linear in S_1 with breakpoints $K_1, \ldots, K_n$.
Payoff is nonnegative on $[0, \infty)$, if and only if nonnegative at 0 and at all breakpoints and right-derivative at K_n is nonnegative (assume $K_1 \leqslant K_2 \leqslant \cdots \leqslant K_n$).
Formally:

$$\Psi^x(0) \geqslant 0$$
$$\Psi^x(K_j) \geqslant 0, j = 1, \ldots, n$$
$$\Psi^x(K_n + 1) - \Psi^x(K_n) \geqslant 0.$$

Linear program

$$
\begin{aligned}
\min \textstyle\sum_{i=1}^{n} S_0^i x_i & & \\
\textstyle\sum_{i=1}^{n} \Psi_i(0) x_i & \geqslant & 0 \\
\textstyle\sum_{i=1}^{n} \Psi_i(K_j) x_i & \geqslant & 0, j = 1, \dots, n \\
\textstyle\sum_{i=1}^{n} \left(\Psi_i(K_n + 1) - \Psi_i(K_n)\right) x_i & \geqslant & 0.
\end{aligned}
$$

Proposition

There is no type A arbitrage if and only if optimal objective value of LP is at least 0

Proposition

Suppose that there is no type A arbitrage. Then, there is no type B arbitrage if and only if the dual of LP has strictly feasible solution.

Constraint matrix

- $\Psi_i(K_j) = \max\{K_j - K_i, 0\}$
- Constraint matrix A of LP has the form

$$A = \begin{pmatrix} K_2 - K_1 & 0 & 0 & \cdots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Theorem

Let $K_1 < K_2 < \cdots < K_n$ denote strike prices of European call options on the same underlying security with same maturity. There are no arbitrage opportunities if and only if prices S_0^i satisfy

1. $S_0^i > 0$, $i = 1, \ldots, n$
2. $S_0^i > S_0^{i+1}$, $i = 1, \ldots, n-1$
3. $C(K_i) := S_0^i$ defined on $\{K_1, \ldots, K_n\}$ is strictly convex function