Prof. Friedrich Eisenbrand
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Question session: 9.12.09
Discussion: 16.12.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 6

Exercise 6.1
Solve the following integer linear program with Branch & Bound.

$$
\begin{aligned}
\min(1,-2)x \\
\begin{pmatrix}-4 & 6\\ 1 & 1\end{pmatrix} x &\leq \begin{pmatrix}9\\4\end{pmatrix}\\
x &\geq \mathbf{0}\\
x &\in \mathbb{Z}^2
\end{aligned}
$$

You can solve the LP subproblems with any computer algebra system or read optimum LP solutions from a drawing.

Exercise 6.2
We want to move to another city. We have n items which we would like to take. Each item $i \in \{1,\dots,n\}$ has a volume v_i and a weight w_i. We have m boxes, whereby box $j \in \{1,\dots,m\}$ has a volume V_j and can carry items of weight at most W_j.

Each item i that we cannot take with us, must be bought again for a price of c_i.

Introduce suitable decision variables (define their meaning) and state an integer linear program, which distributes the items among the boxes, such that for each box the volume bound and the weight bound are not exceeded and the cost of the items left behind is minimized.

Exercise 6.3
In this exercise we develop a method, which computes a cutting plane, that cuts off optimal non-integer solutions.

Consider an integer linear program of the form

$$
\begin{aligned}
\min c^T x & & \quad (IP)\\
Ax &= b\\
x &\geq \mathbf{0}\\
x &\in \mathbb{Z}^n
\end{aligned}
$$

Suppose we use the simplex algorithm to compute a solution x^* to the LP relaxation (that means at least $Ax^* = b$ and $x^* \geq \mathbf{0}$). Let B be the optimal basis (i.e. $A_B x_B^* = b$, $x_{\bar{B}}^* = \mathbf{0}$). Assume there is index

$i \in B$ with $x_i^* \notin \mathbb{Z}$. Since $Ax = b$ is a feasible equation for any solution x, also

$$x_B + A_B^{-1} A_{\bar{B}} x_{\bar{B}} = \underbrace{A_B^{-1} A_B}_{=I} x_B + A_B^{-1} A_{\bar{B}} x_{\bar{B}} = A_B^{-1} A x = A_B^{-1} b \tag{1}$$

holds for any feasible x. Abbreviate β as the ith entry of $A_B^{-1} b$ and let d be the ith row of $A_B^{-1} A_{\bar{B}}$. Then extracting the ith equation from (1) yields that

$$x_i + \sum_{j \in \bar{B}} d_j x_j = \beta$$

holds for any x with $Ax = b$. The *Gomory cut* is now the inequality $x_i + \sum_{j \in \bar{B}} \lfloor d_j \rfloor x_j \leq \lfloor \beta \rfloor$. Prove the following

i) The Gomory cut inequality holds for any solution x to (IP) (that means for any $x \in \mathbb{Z}_+^n$ with $Ax = b$).

ii) The cut inequality does not hold for the fractional solution $x^* \notin \mathbb{Z}^n$.

iii) An optimum solution to the LP relaxation of

$$\begin{aligned} &\min(1,-2)x \\ &\begin{pmatrix} -4 & 6 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix} \\ &x \geq \mathbf{0} \\ &x \in \mathbb{Z}^2 \end{aligned}$$

is $x^* = (1.5, 2.5)$. Obtain a Gomory cut, which cuts of x^*.

Exercise 6.4
Consider a lock-box problem, where c_{ij} is the cost of assigning region i to a lock-box in region j for $j = 1, \ldots, n$. Suppose we wish to open exactly q lock-boxes where $q \in \{1, \ldots, n\}$ is a given integer.

i) Formulate as an integer program the problem of opening exactly q lock-boxes so as to minimize the total cost of assigning each region to an open lock-box.

ii) Formulate in two different ways the constraint that regions cannot send checks to closed lock-boxes.

iii) For the following data

$$q = 2 \qquad \text{and} \qquad (c_{ij})_{1 \leq i,j \leq 5} = \begin{pmatrix} 0 & 4 & 5 & 8 & 2 \\ 4 & 0 & 3 & 4 & 6 \\ 5 & 3 & 0 & 1 & 7 \\ 8 & 4 & 1 & 0 & 4 \\ 2 & 6 & 7 & 4 & 0 \end{pmatrix}$$

compare the linear programming relaxations of your two formulations in question (ii).