

## Exercises

**Optimization in Finance**

Fall 2008

Sheet 5

**Exercise 5.1**

Consider an investor with an initial wealth of  $W_0$ . At time 0, the investor constructs a portfolio comprising one risk-less asset with return  $R_1$  in the first period and one risky asset with return  $R_1^+$  (with probability 0.5) or  $R_1^-$  otherwise. At the end of the period the investor can rebalance his portfolio. The return in the second period is  $R_2$  for the risk-less asset, while it is  $R_2^+$  or  $R_2^-$  for the risky asset (again with equal probability). State a 2-stage stochastic program that guarantees a non-negative wealth at time 2 and maximizes the expected wealth at time 2. Turn this stochastic program into an ordinary linear program.

**Exercise 5.2**

For our company we expect the following net cash flow

Quarter	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
net cash flow	-100	-500	-100	600 or 650	500 or 550	-200	-600	850 or 900

(meaning that for instance we have to pay 100 units at the beginning of quarter 1, while we get 600 or 650 units at beginning of quarter 4). For quarters Q4, Q5, Q8 the cash inflow is uncertain - each possibility appears independently with equal probability. Initially we have 0 units. There are a few options how to invest/borrow money:

- A 6-month loan available each quarter with an interest of 3.6% (for 6 months)
- At any beginning of a quarter we can invest money for 0.5% per quarter.

Ensure that at Q9, we have payed back all loans. We want to maximize our expected wealth at the beginning of Q9.

Formulate this cash-flow management problem as a linear program.

**Exercise 5.3**

Let  $\lambda > 0$  by a parameter. Suppose for a portfolio  $x$  in case of event  $y \in \mathbb{R}$  we incur a loss of  $y$ . For  $\alpha = 0.95$  and the density function

$$p(y) = \begin{cases} 1 & \text{if } -0.95 \leq y \leq 0 \\ 0.05 \frac{1}{\lambda} & \text{if } 0 < y \leq \lambda \\ 0 & \text{otherwise} \end{cases}$$

compute  $\text{VaR}_\alpha(x)$  and  $\text{CVaR}_\alpha(x)$ .

### Exercise 5.4

Assume a continuous loss distribution. Consider the auxiliary function

$$F_\alpha(x, \gamma) = \gamma + \frac{1}{1 - \alpha} \int_{f(x,y) \geq \gamma} (f(x,y) - \gamma) p(y) dy$$

Show that

1.  $F_\alpha(x, \gamma)$  is a convex function (w.r.t.  $\gamma$ )
2.  $F_\alpha(x, \gamma)$  is minimized for  $\gamma = \text{VaR}_\alpha(x)$ .
3.  $\min_\gamma F_\alpha(x, \gamma) = \text{CVaR}_\alpha(x)$

### Exercise 5.5 - Practical Exercise. Due: 9.12.08

For constructing a portfolio with minimal conditional value-at-risk (CVaR) we choose a set of  $n$  assets. Let  $p_{i,s}$  be the price of asset  $i \in \{1, \dots, n\}$  at time  $s = 0, \dots, S$ . The (normalized) loss of a portfolio  $x \in \mathbb{R}_+^n$  ( $\sum_{i=1}^n x_i = 1$ ) in time period  $s$  is

$$f(x, y_s) = \sum_{i=1}^n x_i \frac{p_{i,s} - p_{i,s+1}}{p_{i,s}}.$$

Compute the expected rate of return  $\mu_i$  of asset  $i$  as arithmetic mean over  $\frac{p_{i,s+1} - p_{i,s}}{p_{i,s}}$ . Let  $R$  be a given return. Then  $\min\{\text{CVaR}_\alpha(x) \mid \mu^T x \geq R, \mathbf{1}^T x = 1, x \geq \mathbf{0}\}$  can be written as

$$\begin{aligned} \min \gamma + \frac{1}{(1 - \alpha)S} \sum_{s=1}^S z_s \\ z_s &\geq f(x, y_s) - \gamma \quad \forall s = 1, \dots, S \\ \sum_{i=1}^n x_i \mu_i &\geq R \\ \sum_{i=1}^n x_i &= 1 \\ z, x &\geq \mathbf{0} \\ \gamma &\in \mathbb{R} \end{aligned}$$

(here  $z_s = \max\{f(x, y_s) - \gamma, 0\}$  is an auxiliary variable). Do the following:

1. You can either use the data available on our homepage or you can choose some assets of your choice.
2. Compute the efficient frontier, which is the graph with  $R$  on the first axis and minimal computed CVaR values on the second axis. (You can solve the underlying linear programs with any software, like Matlab). Use parameter  $\alpha = 0.95$  (or  $\alpha = 0.99$ ).

3. Compare the diversification of portfolios  $x^{MVO}$  and  $x^{CVaR}$  obtained by MVO and CVaR minimization (for example by comparing  $\|x^{MVO} - \frac{1}{n}\mathbf{1}\|_2$  and  $\|x^{CVaR} - \frac{1}{n}\mathbf{1}\|_2$  where  $\mathbf{1} = (1, \dots, 1)$ ).
4. Perform a *backtest*, that means for both, MVO and CVaR minimization do the following: Start with a portfolio of 50.000 CHF using data just from the first 2 time periods (for example 2 months). Then for each time  $i = 3, \dots, S$ , rebalance your portfolio (using data from time  $1, \dots, i$ ). Plot the value of your MVO and CVaR portfolios (w.r.t. the time) together with an index, representing your assets (like Dow Jones, if you have mainly American stocks).
5. If you change the *value* invested into asset  $i$  from  $y_{s-1}^i$  to  $y_s^i$  (in time  $s$ ) this causes transaction fees of  $\sqrt{|y_{s-1}^i - y_s^i|}$ . Compute (and compare) the total amount of fees, which you need to pay for MVO and CVaR portfolios.

You can work in groups up to 5 students. The short presentations (max. 5min) of the results take place on 9.12.08. Please prepare a few slides.