Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 11.11.09
Discussion: 18.11.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 4

Note: This is just <u>one</u> way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 4.1
Let $Q \in \mathbb{R}^{n\times n}$ be a positive-definite matrix (i.e. $\forall x \neq \mathbf{0} : x^T Qx > 0$) and $c \in \mathbb{R}^n$ be a vector. Then $\min\{x^T Qx + c^T x \mid x \in \mathbb{R}^n\}$ is bounded, i.e. there exists an M such that $x^T Qx + c^T x \geq -M$ for all $x \in \mathbb{R}^n$.

Solution:
Let $B = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ be the unit ball ($||\cdot||$ gives the Euclidean norm). Define $\delta := \inf_{x\in B}\{x^T Qx\}$. Since B is bounded and closed, B is also compact. Furthermore $x^T Qx$ is a continous function. Then the infimum must be attained at a point, say y, i.e. $||y|| = 1$ and $\delta = y^T Qy > 0$. Furthermore, let $c_{\max} := \max\{|c_i| \mid i = 1,\dots,n\}$. Let $x \neq \mathbf{0}$, then

$$\begin{aligned}
x^T Qx + c^T x &\geq ||x||^2 \underbrace{\frac{x}{||x||}^T Q \frac{x}{||x||}}_{\geq\delta} + ||x|| \underbrace{c^T \frac{x}{||x||}}_{\geq -c_{\max}} \\
&\geq ||x||^2 \cdot \delta - ||x|| c_{\max} \\
&\geq \delta \cdot \left(||x||^2 - ||x|| \frac{c_{\max}}{\delta}\right) \\
&= \delta \cdot \left(||x||^2 - 2 \cdot \frac{1}{2} \frac{c_{\max}}{\delta} + \left(\frac{c_{\max}}{2\delta}\right)^2\right) - \frac{c_{\max}^2}{4\delta} \\
&= \underbrace{\delta}_{\geq 0} \cdot \underbrace{\left(||x|| - \frac{c_{\max}}{2\delta}\right)^2}_{\geq 0} - \frac{c_{\max}^2}{4\delta} \\
&\geq -\underbrace{\frac{c_{\max}^2}{4\delta}}_{=:M}
\end{aligned}$$

since $\frac{x}{||x||} \in B$.

Exercise 4.2

Let $P \subseteq \mathbb{R}^n$ be a polytope, Q be a symmetric, positive semidefinite matrix, $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := x^T Qx$, $x^* = \operatorname{argmin}\{f(x) \mid x \in P\}$. Define

$$C := \max\left\{(x-y)^T Q(x'-y') \mid x,y,x',y' \in P \cup \{\mathbf{0}\}\right\}^{1}$$

and $h(x) = \frac{f(x)-f(x^*)}{4C}$. Prove that $h(x^{(0)}) \leq 1/4$, where $x^{(0)}$ is an arbitrary point in P.

Solution:

One has

$$f(x^{(0)}) - f(x^*) = {x^{(0)}}^T Qx^{(0)} - \underbrace{{x^*}^T Qx^*}_{\geq 0} \leq (x^{(0)} - \mathbf{0})^T Q(x^{(0)} - \mathbf{0}) \leq C.$$

Exercise 4.3

Let $P \subseteq [-M,M]^n$ be a polytope and $Q \in [-M,M]^{n\times n}$ be a symmetric, positive semidefinite matrix. Give a bound[2] (depending on n,M,ε) on the number of iterations k, that the Frank-Wolfe algorithm needs, to reach a solution $x^{(k)}$ such that $f(x^{(k)}) - f(x^*) \leq \varepsilon$ (with $f(x) := x^T Qx$).

Solution:

First of all

$$C = \max\{(x-y)^T Q(x'-y') \mid x,y,x',y' \in P \cup \{\mathbf{0}\}\} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{(x_i - y_i)}_{\leq 2M} \underbrace{Q_{ij}}_{\leq M} \underbrace{(x'_i - y'_i)}_{\leq 2M} \leq 4n^2M^3$$

Then after $k := \lceil 16n^2M^3/\varepsilon \rceil$ many iterations we have

$$\frac{f(x^{(k)}) - f(x^*)}{4C} \leq \frac{1}{k+3}$$

hence

$$f(x^{(k)}) - f(x^*) \leq \frac{4C}{k+3} \leq \frac{16n^2M^3}{k+3} \leq \varepsilon$$

Exercise 4.4

Let $P \subseteq \mathbb{R}^n$ be a polytope, Q be a symmetric, positive semidefinite matrix, $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := x^T Qx$, $w(y) := \min_{v\in P}\{\nabla f(y)^T (v-y) + f(y)\}$,

$$C := \max\left\{(x-y)^T Q(x'-y') \mid x,y,x',y' \in P \cup \{\mathbf{0}\}\right\}$$

Let $\lambda^* = \frac{f(x^{(k)})-w(x^{(k)})}{2C}$ minimizing $g(\lambda) = f(x^{(k)} + \lambda(y^{(k)} - x^{(k)}))$. Prove that $\lambda^* \in [0,1]$.

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[1] We define C here differently then in the lecture to avoid some technical difficulties

[2] the bound does not need to be the best possible one

Solution:

Recall that $\nabla f(x) = 2x^T Q = 2Qx$. Hence

$$\begin{aligned} f(x^{(k)}) - w(x^{(k)}) &= f(x^{(k)}) - \min_{v \in P}\{\nabla f(x^{(k)})^T (v - x^{(k)}) + f(x^{(k)})\} \\ &= \max_{v \in P} \nabla f(x^{(k)})(x^{(k)} - v) \\ &= 2\max_{v \in P}(x^{(k)} - \mathbf{0})^T Q(x^{(k)} - v) \\ &\leq 2C \end{aligned}$$

Exercise 4.5

Prove that

$$\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} \leq e^{-\frac{1}{2(n+1)}}$$

for all $n \geq 2$.

Solution:

Recall that $1 + x \leq e^x$ for all $x \in \mathbb{R}$. Hence

$$\begin{aligned} \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} &= \underbrace{\left(1 - \frac{1}{n+1}\right)}_{\leq e^{-1/(n+1)}} \left(\underbrace{1 + \frac{1}{n^2-1}}_{\leq e^{1/(n^2-1)}}\right)^{(n-1)/2} \\ &\leq \exp\left(-\frac{1}{n+1} + \frac{n-1}{2(n-1)(n+1)}\right) \\ &= \exp\left(-\frac{1}{n+1} + \frac{1}{2(n+1))}\right) \\ &= \exp\left(-\frac{1}{2(n+1)}\right) \end{aligned}$$
