Exercises

Optimization Methods in Finance

Fall 2009

Sheet 3

**Note:** This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

**Exercise 3.1**
Suppose we are given assets \( i = 0, \ldots, n \) which are currently (at time 0) priced at \( S_i^0 \). There are scenarios \( \omega_j \) for \( j = 1, \ldots, m \), in scenario \( \omega_j \) asset \( i \) will have a price of \( S_i^1(\omega_j) \) at time 1. Give an LP, for which any optimum solution gives a portfolio \( x \) that provides type-B arbitrage (if such an arbitrage exists).

**Hint:** Recall that an optimum solution to

\[
\min \sum_{i=0}^n S_i^0 \cdot x_i \\
\sum_{i=0}^n S_i^1(\omega_j) \cdot x_i \geq 0 \quad \forall j = 1, \ldots, m \\
x_i \in \mathbb{R} \quad \forall i = 1, \ldots, n
\]

is used to detect type-A arbitrage.

**Solution:**
By definition, a portfolio \( x \) provides type-B arbitrage iff we have a non-negative ingoing cash-flow at time 0 and a non-negative ingoing cash flow at time 1 for any scenario, but for at least one scenario we have a strictly positive ingoing cashflow at time 1. Consider the following LP

\[
\max \sum_{j=1}^m y_j \\
\sum_{i=0}^n S_i^0 \cdot x_i \leq 0 \\
\sum_{i=0}^n S_i^1(\omega_j) \cdot x_i \geq y_j \quad \forall j = 1, \ldots, m \\
y_j \geq 0 \quad \forall j = 1, \ldots, m \\
x_i \in \mathbb{R} \quad \forall i = 1, \ldots, n
\]

Clearly in an optimum solution one has \( \sum_{i=0}^n S_i^0(\omega_j) = y_j \) (otherwise the \( y_j \)'s could be increased). In other words, \( y_j \) gives the profit in scenario \( \omega_j \). If \( \sum_{j=1}^m y_j > 0 \) then there must be at least one \( j^* \) with \( y_{j^*} > 0 \) (and \( y_j \geq 0 \) for all other \( j \)). Vice versa a portfolio with type-B arbitrage yields a feasible solution with positive objective function value.
Exercise 3.2
Consider the Mean Variance Optimization problem
\[
\begin{align*}
\max & \mu^T x \\
x^T Q x & \leq \sigma^2 \\
\sum_{i=1}^{n} x_i & = 1 \\
x & \geq 0
\end{align*}
\]
where \( \mu_i \) gives the expected return of asset \( i \) and \( Q \) is the covariance matrix. \( \sigma^2 \) is a given parameter, upper-bounding the variance. \( x_i \) gives the ratio, which we are going to invest into asset \( i \).
Suppose we already have a portfolio \( y \) (i.e. \( y \in \mathbb{R}^n \) and \( \sum_{i=1}^{n} y_i = 1 \)). Increasing the ratio \( y_i \), invested into asset \( i \) by some arbitrary \( \delta \in [0, 1] \), costs \( \delta \cdot c_i^+ \geq 0 \), whereby decreasing this ratio by \( \delta \) costs \( \delta \cdot c_i^- \geq 0 \).
Extend the above Mean Variance Optimization problem, such that the expected return minus the arising transaction costs is maximized (this has to be modeled with linear inequalities/equations). Explain the meaning of newly introduced decision variables.

Solution:
Introduce \( \delta_i^+ \) as a variable, defining the increase of portfolio \( i \) and \( \delta_i^- \) the decrease of portfolio \( i \).
\[
\begin{align*}
\max & \mu^T x - \sum_{i=1}^{n} c_i^+ \delta_i^+ - \sum_{i=1}^{n} c_i^- \delta_i^- \\
x^T Q x & \leq \sigma^2 \\
\sum_{i=1}^{n} x_i & = 1 \\
\delta_i^+ & \geq x_i - y_i \ \forall i = 1, \ldots, n \\
\delta_i^- & \geq y_i - x_i \ \forall i = 1, \ldots, n \\
x & \geq 0 \\
\delta_i^+, \delta_i^- & \geq 0 \ \forall i = 1, \ldots, n
\end{align*}
\]
Since \( c_i^+, c_i^- \geq 0 \) in an optimum solution we would have \( \delta_i^+ = \max\{x_i - y_i, 0\} \) and \( \delta_i^- = \max\{y_i - x_i, 0\} \), thus the program is correct.

Exercise 3.3
Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a convex function and \( x, y \in \mathbb{R}^n \). Prove that \( g : [0, 1] \to \mathbb{R} \) with \( g(t) = f(t x + (1-t) y) \) is convex as well.
Solution:
Let \( t_1, t_2, \lambda \in [0, 1] \). Then
\[
g(\lambda t_1 + (1 - \lambda) t_2) = f((\lambda t_1 + (1 - \lambda) t_2) \cdot x + (1 - (\lambda t_1 + (1 - \lambda) t_2)) \cdot y)
\]
\[
= f((\lambda t_1 + (1 - \lambda) t_2) \cdot x + (1 - \lambda t_1 + (1 - \lambda)(1 - t_2)) \cdot y)
\]
\[
= f(\lambda (t_1 x + (1 - t_1)) y + (1 - \lambda) \cdot (t_2 x + (1 - t_2)) y)
\]
\[
f \text{ convex} \leq \lambda \cdot f(t_1 x + (1 - t_1) y) + (1 - \lambda) \cdot f(t_2 x + (1 - t_2) y)
\]
\[
= \lambda \cdot g(t_1) + (1 - \lambda) \cdot g(t_2)
\]

Exercise 3.4
Let \( Q \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Show that \( Q \) is positive semidefinite (i.e. \( \forall x \in \mathbb{R}^n : x^T Q x \geq 0 \)) if and only if all eigenvalues of \( Q \) are non-negative.

Hint: You may use the following theorem from linear algebra: Given a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), there are eigenvalues \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) with eigenvectors \( v_1, \ldots, v_n \in \mathbb{R}^n \) (i.e. \( Av_i = \lambda_i v_i \) for \( i = 1, \ldots, n \)), which form an orthonormal basis of the \( \mathbb{R}^n \) (that means \( v_i^T v_j = 0 \) for all \( i \neq j \) and \( v_i^T v_i = 1 \) for all \( i = 1, \ldots, n \)).

Solution:
Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) be the eigenvalues with orthonormal eigenvectors \( v_1, \ldots, v_n \) according to the above theorem.

\[ \Rightarrow \] Suppose there is an \( i \in \{1, \ldots, n\} \) with \( \lambda_i < 0 \), then \( v_i^T Q v_i = v_i^T (\lambda_i v_i) = \lambda_i v_i^T v_i < 0 \) thus \( Q \) is not positive-semidefinite.

\[ \Leftarrow \] Suppose \( \lambda_1, \ldots, \lambda_n \geq 0 \). Let \( x \in \mathbb{R}^n \). Since \( v_1, \ldots, v_n \) are a basis, we can write
\[
x = \mu_1 v_1 + \ldots + \mu_n v_n
\]
But then
\[
x^T Q x = (\mu_1 v_1 + \ldots + \mu_n v_n)^T Q (\mu_1 v_1 + \ldots + \mu_n v_n)
\]
\[
= \sum_{i=1}^n \sum_{j=1}^n \mu_i \mu_j v_i^T (Q v_j) = \lambda_i v_i^T v_i \geq 0
\]
\[
\geq 0 \quad \text{if } i \neq j \text{ and } 1 \text{ otherwise}
\]
using that \( v_i \perp v_j \) for \( i \neq j \).