Exercises

Optimization Methods in Finance

Fall 2009

Sheet 3

Exercise 3.1
Suppose we are given assets $i = 0, \ldots, n$ which are currently (at time 0) priced at $S^i_0$. There are scenarios $\omega_j$ for $j = 1, \ldots, m$, in scenario $\omega_j$ asset $i$ will have a price of $S^i_1(\omega_j)$ at time 1. Give an LP, for which any optimum solution gives a portfolio $x$ that provides type-B arbitrage (if such an arbitrage exists).

**Hint:** Recall that an optimum solution to
\[
\min \sum_{i=0}^{n} S^i_0 \cdot x_i
\]
\[
\sum_{i=0}^{n} S^i_1(\omega_j) \cdot x_i \geq 0 \quad \forall j = 1, \ldots, m
\]
\[
x_i \in \mathbb{R} \quad \forall i = 0, \ldots, n
\]
is used to detect type-A arbitrage.

Exercise 3.2
Consider the Mean Variance Optimization problem
\[
\max \mu^T x
\]
\[
x^T Q x \leq \sigma^2
\]
\[
\sum_{i=1}^{n} x_i = 1
\]
\[
x \geq 0
\]
where $\mu_i$ gives the expected return of asset $i$ and $Q$ is the covariance matrix. $\sigma^2$ is a given parameter, upper-bounding the variance. $x_i$ gives the ratio, which we are going to invest into asset $i$.

Suppose we already have a portfolio $y$ (i.e. $y \in \mathbb{R}^n_+$ and $\sum_{i=1}^{n} y_i = 1$). Increasing the ratio $y_i$, invested into asset $i$ by some arbitrary $\delta \in [0, 1]$, costs $\delta \cdot c^+_i \geq 0$, whereby decreasing this ratio by $\delta$ costs $\delta \cdot c^-_i \geq 0$.

Extend the above Mean Variance Optimization problem, such that the expected return minus the arising transaction costs is maximized (this has to be modeled with linear inequalities/equations). Explain the meaning of newly introduced decision variables.
Exercise 3.3
Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a convex function and \( x, y \in \mathbb{R}^n \). Prove that \( g : [0, 1] \to \mathbb{R} \) with \( g(t) = f(tx + (1-t)y) \) is convex as well.

Exercise 3.4
Let \( Q \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Show that \( Q \) is positive semidefinite (i.e. \( \forall x \in \mathbb{R}^n : x^T Q x \geq 0 \)) if and only if all eigenvalues of \( Q \) are non-negative.

**Hint:** You may use the following theorem from linear algebra: *Given a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), there are eigenvalues \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) with eigenvectors \( v_1, \ldots, v_n \in \mathbb{R}^n \) (i.e. \( Av_i = \lambda_i v_i \) for \( i = 1, \ldots, n \)), which form an orthonormal basis of the \( \mathbb{R}^n \) (that means \( v_i v_j = 0 \) for all \( i \neq j \) and \( v_i v_i = 1 \) for all \( i = 1, \ldots, n \)).*