Exercises

Optimization in Finance

Fall 2008

Sheet 2

Exercise 2.1
Solve the following linear program by using the simplex method

$$\begin{aligned}
\min \begin{pmatrix} -4 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}^T x & & \\
\begin{pmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} x &=& \begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix} \\
x &\geq& \mathbf{0}
\end{aligned}$$

Start with the basis $B = \{4,5,6\}$. For each iteration give the basis B, the vector x, the objective function value $c^T x$, the inverse A_B^{-1} as well as the vector $\overline{c}$ of reduced costs, the index j which leaves the basis, and the j-th basis direction (your index j may be any index with $\overline{c}_j < 0$).

You may use Matlab or any other computer algebra system to invert A_B and to apply vector multiplication $-$ you don't have to do it by hand.

Exercise 2.2
Let B be an optimal basis of the LP

$$LP: \quad \min\{c^T x \mid Ax = b;\ x \geq \mathbf{0}\}$$

and a feasible basis of

$$LP': \quad \min\{c^T x \mid Ax = b';\ x \geq \mathbf{0}\}$$

Show that then B is also an optimal basis for LP'.

Exercise 2.3
We know that the dual of

$$(P) \quad \min\{c^T x \mid Ax = b;\ x \geq \mathbf{0}\}$$

is

$$(D) \quad \max\{b^T x \mid A^T y \leq c\}.$$

Use this fact to obtain the dual LP for

$$\min\{c^T x \mid Ax \geq b,\ x \geq \mathbf{0}\}$$

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Exercise 2.4
Consider
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
and define $C_i = \{x \in \mathbb{R}^2 \mid x^T Q_i x \leq 1\}$. Then for $i = 1,2$

1. Determine the eigenvalues of Q_i.
2. Is Q_i positive semidefinite?
3. Draw C_i.
4. Is C_i convex? (just say yes or no, you don't need to proof your answer)

Exercise 2.5
Show that

1. If sets $A,B \subseteq \mathbb{R}^n$ are both convex, then $A \cap B$ is also convex.
2. Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric, positive semidefinite matrix. Let $\beta \in \mathbb{R}$, then the set
$$C := \{x \in \mathbb{R}^n \mid x^T Qx \leq \beta\}$$
is convex.
Hint: Remember Cauchy-Schwarz inequality.
3. Use (1) and (2) to conclude that the set of feasible solutions of
$$\begin{array}{rcl} \min \sum_{i=1}^{n} \mu_i x_i & & \\ x^T Qx & \leq & \beta \\ \sum_{i=1}^{n} x_i & = & 1 \\ x & \geq & 0 \end{array}$$
is convex.

The following theorem is known from linear algebra:

Theorem. *Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, there are eigenvalues $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$ with eigenvectors $v_1,\ldots,v_n \in \mathbb{R}^n$ (i.e. $Av_i = \lambda_i v_i$ for $i = 1,\ldots,n$), which form an orthonormal basis of the $\mathbb{R}^n$ (that means $v_i v_j = 0$ for all $i \neq j$ and $v_i v_i = 1$ for all $i = 1,\ldots,n$).*

Exercise 2.6
Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that Q is positive semidefinite (i.e. $\forall x \in \mathbb{R}^n : x^T Qx \geq 0$) if and only if all eigenvalues of Q are non-negative.
Hint: Use the above theorem.

Exercise 2.7 - Practical work. Presentation: November, 4th
In this exercise you are asked to create a reasonable portfolio of assets. Choose a couple of possible assets (indices like S&P 500, NASDAQ, Dax, Dow Jones, for example and stocks of particular companies, sectors like energy, transportation). Search in the web to obtain historical prices of this assets (for example using `http://finance.google.com`) over the last 24 months, taking one price per month into account. Then use the formulas from the lecture to obtain return values and correlations.

You will then derive an optimization problem of the kind

$$\begin{aligned} \min w^T Qw & & \\ \sum_{i=1}^{N} w_i\bar{r}_i &\geq R \\ \sum_{i=1}^{n} w_i &= 1 \\ Aw &\leq b \end{aligned}$$

which determines the optimal investment strategy (w_i gives the fraction which we are going to invest into asset i). Here R is a return (w.r.t. a one month investment) that you should choose meaningful and $Aw \leq b$ gives you additional constraints).

- We recommend Matlab (more precisely the command `quadprog`) to solve above quadratic program.
- If your portfolio contains a single asset to a large fraction introduce additional constraints to bound this fraction.
- Short sales are allowed but risky. If your portfolio contains a large fraction of short sales, introduce constraints to bound their fraction.
- The Markowitz model is very volatile. Thus randomly perturb the returns $\bar{r}_i$ and among the obtained strategies choose a median strategy.
- You can find an example for computing such a portfolio in the book Optimization Methods in Finance, chapter 8.

You can work in groups up to 5 people, the results will be presented (using slides) in the tutorial at November, 4th.