Exercises

Optimization in Finance

Fall 2008

Sheet 1

Exercise 1.1
Write the following linear program in standard form

\[ \begin{align*}
\text{max } & 4x_1 + x_2 - x_3 \\
& x_1 + 3x_3 \leq 6 \\
& 3x_1 + x_2 + 3x_3 \geq 9 \\
& x_1, x_2 \geq 0 \\
& x_3 \in \mathbb{R}
\end{align*} \]

Exercise 1.2
Draw the feasible region (set of feasible solutions) of the following linear program (with 2 variables)

\[ \begin{align*}
\text{max } & 2x_1 - x_2 \\
& x_1 + x_2 \geq 1 \\
& x_1 - x_2 \leq 0 \\
& 3x_1 + x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{align*} \]

Determine the optimal solution to this problem by inspecting your drawing.

Exercise 1.3
Consider the following linear program

\[ \begin{align*}
\text{min } & 2x_1 + 3x_2 \\
& x_1 + x_2 \geq 5 \\
& x_1 \geq 1 \\
& x_2 \geq 2
\end{align*} \]
Prove that \( x^* = (3, 2) \) is the optimal solution by showing that the objective value of any feasible solution is at least 12 (hint: duality).

**Exercise 1.4**
A polyhedron is a set \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) for \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \). Show that each polyhedron is convex (recall that a set \( C \subseteq \mathbb{R}^n \) is convex if and only if for any \( x, y \in C \) one has \( \lambda x + (1 - \lambda) y \in C \) for all \( \lambda \in [0, 1] \)).

**Exercise 1.5**
The vector \( x^* = (0, 1, 1, 1) \) is an optimal solution of

\[
\begin{align*}
\min \ (1, 1, 0, 2) \cdot x \\
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 0 & 2 \\
2 & 0 & 0 & 0
\end{pmatrix} x &= \begin{pmatrix}
6 \\
3 \\
0
\end{pmatrix} \\
\quad \quad \quad = A x \geq 0
\end{align*}
\]

with \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \). Use the proof of Lemma 1.1 to find another optimal solution \( x' \) such that \( A_{J'} \) has full column rank with \( J' = \{ i \mid x'_i > 0 \} \).

**Exercise 1.6**
For our company we expect the following net cash flow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Quarter} & Q1 & Q2 & Q3 & Q4 & Q5 & Q6 & Q7 & Q8 \\
\hline
\text{net cash flow} & -100 & -500 & -100 & 600 & 500 & -200 & -600 & 900
\end{array}
\]

(meaning that for instance we have to pay 100 units at the beginning of quarter 1, while we get 600 units at beginning of quarter 4). Initially we have 0 units. There are a couple of options how to invest/borrow money:

- A 2-year loan (to borrow money) available only at the beginning of Q1 (has to be payed back at beginning of Q9) for 8% interest (for 2 years).
- A 6-month loan available each quarter with an interest of 3.6% (for 6 months)
- A 3-month loan available each quarter at an interest of 2.5% (per quarter)
- At any beginning of a quarter we can invest money for 0.5% per quarter.

Ensure that at Q9, we have payed back all loans. We want to maximize our wealth at the beginning of Q9.

Formulate this cash-flow management problem as a linear program (introduce suitable decision variables). Then model the LP with ZIMPL and solve it with QSOpt. Interpret the outcome.