Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 8.12.10
Discussion: 15.12.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 6

Exercises marked with **(*)** qualify for one bonus point, if correctly presented in the discussion session

Exercise 6.1 (*)
We consider a combinatorial action in which we have $b_i \in \mathbb{N}$ times item $i \in \{1,\ldots,n\}$, $B := \max_{i=1,\ldots,n}\{b_i\}$ (in other words, there are just n many different undistinguishable items). Bids are pairs (S_j, p_j), i.e. bidder $j \in 1,\ldots,m$ is willing to pay p_j for items $S_j \subseteq \{1,\ldots,n\}$. Design a dynamic programming approach to select a feasible subset of bids that maximize the profit.
Remark: This can be done with $O((B+1)^n)$ table entries.

Exercise 6.2 (*)
Solve the following integer linear program with Branch & Bound.

$$\begin{aligned} \min(1,-2)x & & \\ \begin{pmatrix} -4 & 6 \\ 1 & 1 \end{pmatrix} x &\leq \begin{pmatrix} 9 \\ 4 \end{pmatrix} \\ x &\geq \mathbf{0} \\ x &\in \mathbb{Z}^2 \end{aligned}$$

You can solve the LP subproblems with any computer algebra system or by inspecting a drawing.

Exercise 6.3 (*)
Recall that for a *combinatorial auction*, an actioneer is selling items $M = \{1,\ldots,n\}$ and receives bids $B_j = (S_j, p_j)$ with $j = 1,\ldots,n$, where $S_j \subseteq M$ is a subset of the items and $p_j \geq 0$ is the price. The goal is to select a feasible subset of the bids that maximize the cumulated price. This problem can be formulated as an integer linear program

$$\begin{aligned} \max \sum_{i=1}^{n} x_j p_j & & & \\ \sum_{j:i\in S_j} x_j &\leq 1 & \forall i \in M \\ 0 \leq x_j &\leq 1 & \forall j = 1,\ldots,n \\ x_j &\in \mathbb{Z} & \forall j = 1,\ldots,n \end{aligned}$$

Let IP be its optimum objective function value and let LP be the optimum value of the linear programming relaxation (i.e. the value of the above formulation without the integrality constraint). Of course $LP \geq IP$. Can you find a family of instances, where $\frac{LP}{IP}$ tends to ∞, when n grows?

Remark: The supremum of the ratio $\frac{LP}{IP}$ is usually called the *integrality gap*. A gap of roughly $\sqrt{n}$ is possible, even if all prices are 1.

Exercise 6.4 - Practical exercise - 1 points
Implement the branch and bound algorithm.

1. You can implement the algorithm in one of the programming languages C/C++/Java/Pascal/Basic/Matlab (you can choose your favourite one).

2. Your submission should contain your (compilable) code and both, the optimum integral solution and its value for the 2 problems

$$\begin{array}{lrrrrrrrrrrcll}
\max & (5, & 6, & 10, & 10, & 8, & 10, & 10, & 10, & 9, & 10)^T x & & (IP1) & \\
 & (4788, & 3703, & 8104, & 8357, & 5089, & 6832, & 9723, & 7054, & 3680, & 6088)x & \leq & 10000 & \\
 & & & & & & & & & & -x_i & \leq & 0 & \forall i = 1, \ldots, 10 \\
 & & & & & & & & & & x_i & \in & \mathbb{Z} & \forall i = 1, \ldots, 10
\end{array}$$

$$\begin{array}{ccc}
\max(1.1, 1.2, 1.3, 1.4, 1.5, 1.6)^T x & & (IP2) \\
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} x & \leq & \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
x_1, \ldots, x_6 & \in & \mathbb{Z}
\end{array}$$

3. Send the files till **22.12.10 (23:59h)** to thomas.rothvoss@epfl.ch.

4. You can work in groups up to 3 people (you need only one submission per group).

Hints:

1. You can use any LP library to solve the relaxations (in Matlab, there is a build-in command to solve linear programs, for C/C++/Java there are plenty of libraries available).

2. We recommend to test your instance with a small example, say from exercise 6.2.

3. If for a fractional solution x^* one has several variables $x_i^* \notin \mathbb{Z}$ you can branch on an arbitrary such i.