Exercises

Optimization Methods in Finance

Fall 2010

Sheet 4

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 4.1 (*)
Consider the primal

$$\max\{c^T x \mid Ax \leq b\} \quad (P)$$

and the dual LP.

$$\min\{y^T b \mid y^T A = c^T, y \geq \mathbf{0}\} \quad (D)$$

i) Suppose that (P) is feasible and bounded, say $x^* \in \mathbb{R}^n$ is an optimal solution. Let $I \subseteq \{1,\ldots,m\}$ be the set of active constraints at x^* (i.e. $I = \{i \in \{1,\ldots,m\} \mid A_i x^* = b_i\}$ and A_i denotes the ith row of A). Show that there exists a $y^* \in \mathbb{R}^m$ with

$$y_i^* \geq 0 \; \forall i \in I, \quad y_i^* = 0 \; \forall i \notin I, \quad {y^*}^T A = c^T$$

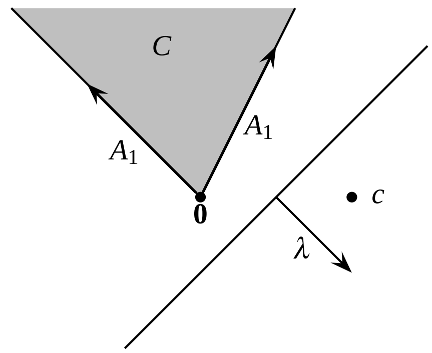
Hint: Assume for contradiction that there is no such y^*, i.e. $c \notin \{\sum_{i\in I} A_i y_i \mid y_i \geq 0\}$ and apply the strict separating hyperplane theorem: *Given a closed convex set C and a point $x_0 \notin C$, there exists a hyperplane $a^T x = \beta$ with $a^T x_0 < \beta$, $a^T x > \beta \; \forall x \in C$.* Then show that x^* would not be optimal.

ii) Show that the vector y^* from $i)$ is an optimal dual solution with objective function value $c^T x^*$.

iii) Suppose that (P) is infeasible and the dual problem (D) is feasible. Show that the dual problem is unbounded.
Hint: Show that there is a $v \in \mathbb{R}^m \backslash \{\mathbf{0}\}$ with $A^T v = 0, v \geq \mathbf{0}, b^T v < 0$.

Solution:

1. Assume for contradiction there is no such y^*, i.e. $c \notin C$ with $C := \{\sum_{i\in I} A_i y_i \mid y_i \geq 0\}$. The set C is convex and bounded, hence there is a strictly separating hyperplane with $\lambda^T c > \beta, \lambda^T x < \beta \forall x \in C$. Then $\lambda A_i \leq 0$ for all i. Furthermore $\mathbf{0} \in C$, hence $\lambda^T c > \beta > 0$.



Let $\delta := \min_{i \notin I}\{b_i - A_i x^*\} > 0$ the minimum slack of inactive constraints. Choose $\varepsilon > 0$ such that $\varepsilon A_i \lambda \leq \delta$. Then

$$\begin{aligned} A_i(x^* + \varepsilon\lambda) &= \underbrace{A_i x^*}_{=b_i} + \varepsilon \underbrace{A_i \lambda}_{\leq 0} \leq b_i \quad \forall i \in I \\ A_i(x^* + \varepsilon\lambda) &\leq (b_i - \delta) + \varepsilon A_i \lambda \leq b_i \quad \forall i \notin I \end{aligned}$$

Furthermore

$$c^T(x^* + \varepsilon\lambda) = c^T x^* + \underbrace{\varepsilon c^T \lambda}_{>0}.$$

A contradiction to the optimality of x^*. Hence there is such a y^*.

2. First of all y^* is a feasible dual solution. Secondly

$$y^{*T} b - \underbrace{c^T}_{=y^T A} x^* = y^{*T} b - y^T A x^* = y^{*T} \cdot (b - Ax^*) = \sum_{j=1}^{m} \underbrace{y_i^* \cdot (b_i - A_i x^*)}_{=0} = 0$$

using that either $A_i x^* = b$ or $y_i^* = 0$.

3. (P) being infeasible means that

$$b \notin \underbrace{\left\{ \sum_{i=1}^{n} A^i x_i + \mu \mid x \in \mathbb{R}^n, \mu \geq \mathbf{0} \right\}}_{\text{set of feasible right hand sides of}} =: K$$

Again there is a strictly separating hyperplane $\lambda^T b < \beta < \lambda^T z$ for all $z \in K$. Then

$$\beta < \lambda^T \left(\sum_{i=1}^{n} A^i x_i + \sum_{j=1}^{m} \mu_j e_j \right) = \sum_{i=1}^{n} \underbrace{\lambda^T A^i}_{\text{has to be } 0} x_i + \sum_{j=1}^{m} \mu_j \underbrace{\lambda^T e_j}_{\text{has to be } \geq 0} \quad \forall x \in \mathbb{R}^n \ \forall \mu \geq \mathbf{0}$$

But then $\lambda^T A = \mathbf{0}$ and $\lambda \geq \mathbf{0}$. Let y be any dual feasible point (which exists by assumption). The $y + \mathbb{R}_+ \lambda$ is a half-line of dual feasible points whose objective function tends to $-\infty$.

Exercise 4.2 (*)
Let x^* be a solution to

$$\min\{c^T x \mid Ax = b, x \geq \mathbf{0}\} \quad (P)$$

and y^* be a feasible solution to

$$\max\{b^T y \mid A^T y \leq c\} \quad (D)$$

Prove that the following conditions are equivalent

1. x^* and y^* are both optimal (i.e. x^* optimal for (P) and y^* optimal for (D))
2. $\forall i : x_i^* > 0 \Rightarrow (c - A^T y^*)_i = 0$

Hint: Recall that by strong duality, the optimal values for (P) and (D) are the same, given that both systems are feasible.

Solution:

- $(2) \Rightarrow (1)$. But conditioning on (2), one has

$$c^T x^* - \underbrace{b^T}_{=(Ax^*)^T} y^* = c^T x^* - (Ax^*)^T y^* = x^{*T}(c - A^T y^*) = \sum_{i=1}^{n} \underbrace{\overbrace{x_i^*}^{\geq 0} \cdot \overbrace{(c - A^T y^*)_i}^{\geq 0}}_{=0 \text{ by } (2)} = 0$$

  In fact, weak duality $c^T x^* \geq b^T y^*$ follows from the fact that this sum can never be negative (even if (2) is not satisfied).

- $(1) \Rightarrow (2)$. If x^* and y^* are both optimal, we have $c^T x^* = b^T y^*$ by strong duality. Then

$$0 = c^T x^* - b^T y^* = \sum_{i=1}^{n} \underbrace{x_i^*}_{\geq 0} \cdot \underbrace{(c - A^T y^*)_i}_{\geq 0}$$

  If there is any i with $x_i^* > 0$ and $(c - A^T y^*)_i > 0$, then this sum couldn't be 0.

Exercise 4.3 (*)
Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 40 CHF:

Option i	strike price K_i (in CHF)	price S_0^i (in CHF)
1	30	10
2	40	7
3	50	10/3
4	60	0

Construct a portfolio of the above options that provides a type-A arbitrage opportunity.
Hint: You may use any LP solver.

Solution:

Recall that a European Call option with price p and strike price c means that we can buy for a price of p at time 0 the right to buy the underlying asset for a price c at time 1. Let x_i be the amount of options i that we buy ($x_i < 0$ means we sell $|x_i|$ times option i). The LP to detect type-A arbitrage is (in general for n European Call options)

$$\min \sum x_i S_0^i \sum_{i=1}^{n} x_i \max\{S_1 - K_i, 0\} \quad \geq \quad 0 \quad \forall S_1 \geq 0 x \in \mathbb{R}^n$$

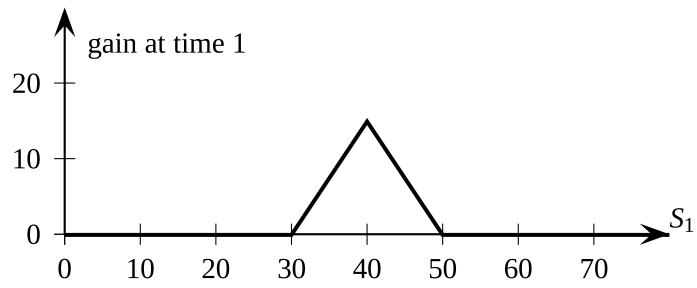
This LP has an infinite number of constraints. Fortunately we saw in the lecture, that $\sum_{i=1}^{n} x_i \max\{S_1 - K_i, 0\}$ is a piecewise linear function in S_1 with breakpoints $K_1, \dots, K_n$, hence it suffices to keep the constraints for S_1 being one of those breakpoints (plus one constraint that ensures that the slope for $S_1 > \max K_i$ is positive. Hence we end up with the following LP

$$\begin{aligned}
&\min \sum_{i=1}^{4} S_0^i x_i \\
\sum_{i=1}^{4} \max\{K_1 - K_i, 0\} x_i &\geq 0 \quad (\text{for } S_1 = K_1) \\
\sum_{i=1}^{4} \max\{K_2 - K_i, 0\} x_i &\geq 0 \quad (\text{for } S_1 = K_2) \\
\sum_{i=1}^{4} \max\{K_3 - K_i, 0\} x_i &\geq 0 \quad (\text{for } S_1 = K_3) \\
\sum_{i=1}^{4} \max\{K_4 - K_i, 0\} x_i &\geq 0 \quad (\text{for } S_1 = K_4) \\
\sum_{i=1}^{4} (\max\{K_4 - K_i + 1, 0\} - \max\{K_4 - K_i, 0\}) x_i &\geq 0 \quad (\text{for } S_1 = K_1) \\
x_i &\in \mathbb{R}
\end{aligned}$$

which is

$$\begin{array}{lrrrrcl}
\min & 10x_1 & +7x_2 & +\frac{10}{3}x_3 & +0x_4 & & \\
 & 10x_1 & & & & \geq & 0 \\
 & 20x_1 & +10x_2 & & & \geq & 0 \\
 & 30x_1 & +20x_2 & +10x_3 & & \geq & 0 \\
 & x_1 & +x_2 & +x_3 & +x_4 & \geq & 0 \\
 & x_1, & x_2, & x_3, & x_4 & \in & \mathbb{R}
\end{array}$$

(and $10x_1 + 7x_2 + \frac{10}{3}x_3 + 0x_4 = -1$ for normalization). Note that the constraint for $S_1 = K_1$ is "$0 \geq 0$" and can be omitted. We obtain a (not unique) solution $x = (1.5, -3, 1.5, 0)$ giving a negative objective function value (namely -1). Depending on the price S_1 of the underlying asset at time 1 we furthermore earn the following amount at time 1 (additionally to the 1 CHF that we got at time 0):

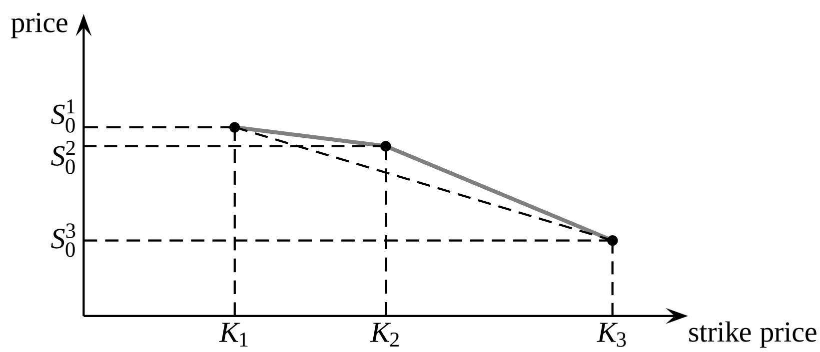


Exercise 4.4 (*)
Suppose we are given 3 European Call options (all w.r.t. the same underlying asset, all with the same maturity), Option i with a price of S_0^i and strike price of K_i. Suppose that $K_1 < K_2 < K_3; S_0^1 > S_0^2 > S_0^3$ and the point (K_2, S_0^2) lies above (or on) the line segment that connects (K_1, S_0^1) and (K_3, S_0^3). Formally there is a $0 < \lambda < 1$ with $K_2 = \lambda K_1 + (1-\lambda)K_3$ and

$$S_0^2 \geq \lambda S_0^1 + (1-\lambda)S_0^3.$$

Give an *explicit* formula for a portfolio that provides arbitrage. Which type of arbitrage is it?

Solution:
The situation can be depicted as follows:

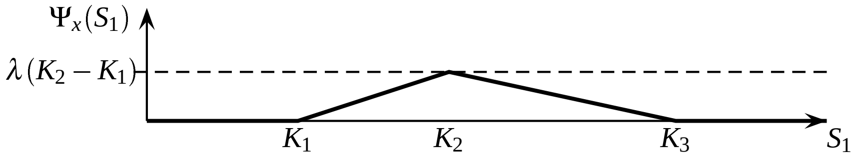


We choose a portfolio $x \in \mathbb{R}^3$ with $x_1 = \lambda, x_2 = -1, x_3 = (1-\lambda)$. Then $\sum_{i=1}^3 S_0^i x_i = S_0^1\lambda - S_0^2 + (1-\lambda)S_0^3 \leq 0$ by assumption, hence we have a non-negative ingoing cash-flow at time 0. On the other hand, let us consider the gain at time 1

$$\Psi_x(S_1) = \sum_{i=1}^{3} \max\{S_1 - K_i, 0\} \cdot x_i$$

depending on the price S_1 which the asset reaches. We verify that $\forall S_1 \geq 0 : \Psi_x(S_1) \geq 0$ and $\exists S_1 \geq 0 : \Psi_x(S_1) > 0$:

- $S_1 = K_1 : \Psi_x(K_1) = 0$
- $S_1 = K_2 : \Psi_x(K_2) = \lambda(K_2 - K_1) > 0$
- $S_1 = K_3 : \Psi_x(K_3) = \lambda(K_3 - K_1) - (K_3 - K_2) = -(\lambda K_1 + (1-\lambda)K_3) + K_2 = 0$
- $S_1 \to \infty : \Psi_x(K_3 + 1) - \Psi_x(K_3) = x_1 + x_2 + x_3 = 0$



In other words, the payoff at time 1 is never negative and for $S_1 \in]K_1, K_3[$ it is strictly positive. Hence the portfolio x provides a type-B arbitrage. If the point (K_2, S_0^2) lies *strictly* above the line segment connecting (K_1, S_0^1) and (K_3, S_0^3), then x additionally provides type-A arbitrage since then $S_0^1\lambda - S_0^2 + (1-\lambda)S_0^3 < 0$, hence the ingoing cash flow at time 0 would be strictly positive.
