Exercises

Optimization Methods in Finance

Fall 2010

Sheet 4

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 4.1 (*)
Consider the primal
\[ \max \{ c^T x \mid Ax \leq b \} \quad (P) \]
and the dual LP.
\[ \min \{ y^T b \mid y^T A = c^T, y \geq 0 \} \quad (D) \]

i) Suppose that \((P)\) is feasible and bounded, say \(x^* \in \mathbb{R}^n\) is an optimal solution. Let \(I \subseteq \{1, \ldots, m\}\) be the set of active constraints at \(x^*\) (i.e. \(I = \{i \in \{1, \ldots, m\} \mid A_i x^* = b_i\}\) and \(A_i\) denotes the \(i\)th row of \(A\)). Show that there exists a \(y^* \in \mathbb{R}^m\) with
\[
y^*_i \geq 0 \forall i \in I, \quad y^*_i = 0 \forall i \notin I, \quad y^{*T} A = c^T
\]

Hint: Assume for contradiction that there is no such \(y^*\), i.e. \(c \not\in \{\sum_{i \in I} A_i y_i \mid y_i \geq 0\}\) and apply the strict separating hyperplane theorem: *Given a closed convex set \(C\) and a point \(x_0 \not\in C\), there exists a hyperplane \(a^T x = \beta\) with \(a^T x_0 < \beta\), \(a^T x > \beta \forall x \in C\). Then show that \(x^*\) would not be optimal.*

ii) Show that the vector \(y^*\) from i) is an optimal dual solution with objective function value \(c^T x^*\).

iii) Suppose that \((P)\) is infeasible and the dual problem \((D)\) is feasible. Show that the dual problem is unbounded.

Hint: Show that there is a \(v \in \mathbb{R}^m \setminus \{0\}\) with \(A^T v = 0, v \geq 0, b^T v < 0\).

Solution:

1. Assume for contradiction there is no such \(y^*\), i.e. \(c \not\in C\) with \(C := \{\sum_{i \in I} A_i y_i \mid y_i \geq 0\}\). The set \(C\) is convex and bounded, hence there is a strictly separating hyperplane with \(\lambda^T c > \beta\), \(\lambda^T x < \beta \forall x \in C\). Then \(\lambda A_i \leq 0\) for all \(i\). Furthermore \(0 \in C\), hence \(\lambda^T c > \beta > 0\).
Let $\delta := \min_{i \in I} \{ b_i - A_i x^* \} > 0$ the minimum slack of inactive constraints. Choose $\varepsilon > 0$ such that $\varepsilon A_i \lambda \leq \delta$. Then

$$A_i(x^* + \varepsilon \lambda) = A_i x^* + \varepsilon A_i \lambda \leq b_i \quad \forall i \in I$$

$$A_i(x^* + \varepsilon \lambda) \leq (b_i - \delta) + \varepsilon A_i \lambda \leq b_i \quad \forall i \notin I$$

Furthermore

$$c^T (x^* + \varepsilon \lambda) = c^T x^* + \varepsilon c^T \lambda > 0.$$

A contradiction to the optimality of $x^*$. Hence there is such a $y^*$.

2. First of all $y^*$ is a feasible dual solution. Secondly

$$y^T b - c^T x^* = y^T b - y^T A x^* = y^T b - (b - A x^*) = \sum_{j=1}^{m} y_i^T \cdot (b_i - A_i x^*) = 0$$

using that either $A_i x^* = b$ or $y_i^* = 0$.

3. ($P$) being infeasible means that

$$b \notin \left\{ \sum_{i=1}^{n} A'_i x_i + \mu \mid x \in \mathbb{R}^n, \mu \geq 0 \right\} =: K$$

Again there is a strictly separating hyperplane $\lambda^T b < \beta < \lambda^T z$ for all $z \in K$. Then

$$\beta < \lambda^T \left( \sum_{i=1}^{n} A'_i x_i + \sum_{j=1}^{m} \mu_j e_j \right) = \sum_{i=1}^{n} \lambda_i^T A'_i x_i + \sum_{j=1}^{m} \mu_j \lambda_j^T e_j \quad \forall x \in \mathbb{R}^n \forall \mu \geq 0$$

But then $\lambda^T A = 0$ and $\lambda \geq 0$. Let $y$ be any dual feasible point (which exists by assumption). The $y + \mathbb{R}_+ \lambda$ is a half-line of dual feasible points whose objective function tends to $-\infty$.
Exercise 4.2 (*)
Let $x^*$ be a solution to
$$\min \{c^T x \mid Ax = b, x \geq 0\} \quad (P)$$
and $y^*$ be a feasible solution to
$$\max \{b^T y \mid A^T y \leq c\} \quad (D)$$
Prove that the following conditions are equivalent
1. $x^*$ and $y^*$ are both optimal (i.e. $x^*$ optimal for (P) and $y^*$ optimal for (D))
2. $\forall i : x_i^* > 0 \Rightarrow (c - A^T y^*)_i = 0$

**Hint:** Recall that by strong duality, the optimal values for (P) and (D) are the same, given that both systems are feasible.

**Solution:**
- (2) $\Rightarrow$ (1). But conditioning on (2), one has
  $$c^T x^* - b^T y^* = c^T x^* - (A x^*)^T y^* = x^T (c - A^T y^*) = \sum_{i=1}^{n} x_i^* \cdot (c - A^T y^*)_i = 0$$
  In fact, weak duality $c^T x^* \geq b^T y^*$ follows from the fact that this sum can never be negative (even if (2) is not satisfied).
- (1) $\Rightarrow$ (2). If $x^*$ and $y^*$ are both optimal, we have $c^T x^* = b^T y^*$ by strong duality. Then
  $$0 = c^T x^* - b^T y^* = \sum_{i=1}^{n} x_i^* \cdot (c - A^T y^*)_i$$
  If there is any $i$ with $x_i^* > 0$ and $(c - A^T y^*)_i > 0$, then this sum couldn’t be 0.

Exercise 4.3 (*)
Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 40 CHF:

<table>
<thead>
<tr>
<th>Option $i$</th>
<th>strike price $K_i$ (in CHF)</th>
<th>price $S_{i0}$ (in CHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>10/3</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>0</td>
</tr>
</tbody>
</table>

Construct a portfolio of the above options that provides a type-A arbitrage opportunity.

**Hint:** You may use any LP solver.
Solution:

Recall that a European Call option with price $p$ and strike price $c$ means that we can buy for a price of $p$ at time 0 the right to buy the underlying asset for a price $c$ at time 1. Let $x_i$ be the amount of options $i$ that we buy ($x_i < 0$ means we sell $|x_i|$ times option $i$). The LP to detect type-A arbitrage is (in general for $n$ European Call options)

$$\min \sum x_i S_i^0 \sum_{i=1}^n x_i \max\{S_i - K_i, 0\} \geq 0 \ \forall S_1 \geq 0 x \in \mathbb{R}^n$$

This LP has an infinite number of constraints. Fortunately we saw in the lecture, that $\sum_{i=1}^n x_i \max\{S_i - K_i, 0\}$ is a piecewise linear function in $S_1$ with breakpoints $K_1, \ldots, K_n$, hence it suffices to keep the constraints for $S_1$ being one of those breakpoints (plus one constraint that ensures that the slope for $S_1 > \max K_i$ is positive. Hence we end up with the following LP

$$\min \sum_{i=1}^4 S_i^0 x_i$$

$$\sum_{i=1}^4 \max\{K_i - K_i, 0\} x_i \geq 0 \text{ (for } S_1 = K_1)$$

$$\sum_{i=1}^4 \max\{K_2 - K_i, 0\} x_i \geq 0 \text{ (for } S_1 = K_2)$$

$$\sum_{i=1}^4 \max\{K_3 - K_i, 0\} x_i \geq 0 \text{ (for } S_1 = K_3)$$

$$\sum_{i=1}^4 \max\{K_4 - K_i, 0\} x_i \geq 0 \text{ (for } S_1 = K_4)$$

$$\sum_{i=1}^4 (\max\{K_4 - K_i + 1, 0\} - \max\{K_4 - K_i, 0\}) x_i \geq 0 \text{ (for } S_1 = K_1)$$

$$x_i \in \mathbb{R}$$

which is

$$\min \begin{pmatrix} 10x_1 + 7x_2 + 10x_3 + 0x_4 \\ 10x_1 \geq 0 \\ 20x_1 + 10x_2 \geq 0 \\ 30x_1 + 20x_2 + 10x_3 \geq 0 \\ x_1 + x_2 + x_3 + x_4 \geq 0 \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{pmatrix}$$

(and $10x_1 + 7x_2 + 10x_3 + 0x_4 = -1$ for normalization). Note that the constraint for $S_1 = K_1$ is “$0 \geq 0$” and can be omitted. We obtain a (not unique) solution $x = (1.5, -3, 1.5, 0)$ giving a negative objective function value (namely $-1$). Depending on the price $S_1$ of the underlying asset at time 1 we furthermore earn the following amount at time 1 (additionally to the 1 CHF that we got at time 0):
Exercise 4.4 (*)

Suppose we are given 3 European Call options (all w.r.t. the same underlying asset, all with the same maturity), Option \( i \) with a price of \( S_0^i \) and strike price of \( K_i \). Suppose that \( K_1 < K_2 < K_3; S_0^1 > S_0^2 > S_0^3 \) and the point \( (K_2, S_0^2) \) lies above (or on) the line segment that connects \( (K_1, S_0^1) \) and \( (K_3, S_0^3) \). Formally there is a \( 0 < \lambda < 1 \) with

\[
S_0^2 \geq \lambda S_0^1 + (1 - \lambda) S_0^3.
\]

Give an explicit formula for a portfolio that provides arbitrage. Which type of arbitrage is it?

Solution:
The situation can be depicted as follows:

We choose a portfolio \( x \in \mathbb{R}^3 \) with \( x_1 = \lambda, x_2 = -1, x_3 = (1 - \lambda) \). Then \( \sum_{i=1}^3 S_0^i x_i = S_0^1 \lambda - S_0^2 + (1 - \lambda) S_0^3 \leq 0 \) by assumption, hence we have a non-negative ingoing cash-flow at time 0. On the other hand, let us consider the gain at time 1

\[
\Psi_x(S_1) = \sum_{i=1}^3 \max\{S_1 - K_i, 0\} \cdot x_i
\]

depending on the price \( S_1 \) which the asset reaches. We verify that \( \forall S_1 \geq 0 : \Psi_x(S_1) \geq 0 \) and \( \exists S_1 \geq 0 : \Psi_x(S_1) > 0 \):

- \( S_1 = K_1 : \Psi_x(K_1) = 0 \)
- \( S_1 = K_2 : \Psi_x(K_2) = \lambda (K_2 - K_1) > 0 \)
- \( S_1 = K_3 : \Psi_x(K_3) = \lambda (K_3 - K_1) - (K_3 - K_2) = -(\lambda K_1 + (1 - \lambda) K_3) + K_2 = 0 \)
- \( S_1 \to \infty : \Psi_x(K_3 + 1) - \Psi_x(K_3) = x_1 + x_2 + x_3 = 0 \)
In other words, the payoff at time 1 is never negative and for $S_1 \in [K_1, K_3]$ it is strictly positive. Hence the portfolio $x$ provides a type-B arbitrage. If the point $(K_2, S_0^2)$ lies strictly above the line segment connecting $(K_1, S_0^1)$ and $(K_3, S_0^3)$, then $x$ additionally provides type-A arbitrage since then $S_0^1 \lambda - S_0^2 + (1 - \lambda) S_0^3 < 0$, hence the ingoing cash flow at time 0 would be strictly positive.