Exercises

Optimization Methods in Finance

Fall 2010
Sheet 4

Exercises marked with (*) qualify for one bonus point, if correctly presented in the discussion session.

Exercise 4.1 (*)
Consider the primal

$$\max \{ c^T x \mid Ax \leq b \} \quad (P)$$

and the dual LP.

$$\min \{ y^T b \mid y^T A = c^T, y \geq 0 \} \quad (D)$$

i) Suppose that \((P)\) is feasible and bounded, say \(x^* \in \mathbb{R}^n\) is an optimal solution. Let \(I \subseteq \{1, \ldots, m\}\) be the set of active constraints at \(x^*\) (i.e. \(I = \{ i \in \{1, \ldots, m\} \mid A_i x^* = b_i \}\) and \(A_i\) denotes the \(i\)th row of \(A\)). Show that there exists a \(y^* \in \mathbb{R}^m\) with

\[
y^*_i \geq 0 \forall i \in I, \quad y^*_i = 0 \forall i \notin I, \quad y^* A = c^T
\]

**Hint:** Assume for contradiction that there is no such \(y^*\), i.e. \(c \notin \{ \sum_{i \in I} A_i y_i \mid z_i \geq 0 \}\) and apply the strict separating hyperplane theorem: Given a closed convex set \(C\) and a point \(x_0 \notin C\), there exists a hyperplane \(a^T x = \beta\) with \(a^T x_0 < \beta\), \(a^T x > \beta \forall x \in C\). Then show that \(x^*\) would not be optimal.

ii) Show that the vector \(y^*\) from i) is an optimal dual solution with objective function value \(c^T x^*\).

iii) Suppose that \((P)\) is infeasible and the dual problem \((D)\) is feasible. Show that the dual problem is unbounded.

**Hint:** Show that there is a \(v \in \mathbb{R}^m \setminus \{0\}\) with \(A^T v = 0, v \geq 0, b^T v < 0\).

Exercise 4.2 (*)
Let \(x^*\) be a solution to

$$\min \{ c^T x \mid Ax = b, x \geq 0 \} \quad (P)$$

and \(y^*\) be a feasible solution to

$$\max \{ b^T y \mid A^T y \leq c \} \quad (D)$$

Prove that the following conditions are equivalent

1. \(x^*\) and \(y^*\) are both optimal (i.e. \(x^*\) optimal for \((P)\) and \(y^*\) optimal for \((D)\))

2. \(\forall i : x^*_i > 0 \Rightarrow (c - A^T y^*)_i = 0\)
**Hint:** Recall that by strong duality, the optimal values for \((P)\) and \((D)\) are the same, given that both systems are feasible.

**Exercise 4.3 (*)**
Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 40 CHF:

<table>
<thead>
<tr>
<th>Option</th>
<th>Strike price (K_i) (in CHF)</th>
<th>Price (S^i_0) (in CHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>10/3</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>0</td>
</tr>
</tbody>
</table>

Construct a portfolio of the above options that provides a type-A arbitrage opportunity.

**Hint:** You may use any LP solver.

**Exercise 4.4 (*)**
Suppose we are given 3 European Call options (all w.r.t. the same underlying asset, all with the same maturity), Option \(i\) with a price of \(S^i_0\) and strike price of \(K_i\). Suppose that \(K_1 < K_2 < K_3; S^1_0 > S^2_0 > S^3_0\) and the point \((K_2, S^2_0)\) lies above (or on) the line segment that connects \((K_1, S^1_0)\) and \((K_3, S^3_0)\). Formally there is a \(0 < \lambda < 1\) with \(K_2 = \lambda K_1 + (1 - \lambda) K_3\) and

\[
S^2_0 \geq \lambda S^1_0 + (1 - \lambda) S^3_0.
\]

Give an *explicit* formula for a portfolio that provides arbitrage. Which type of arbitrage is it?