Exercises
Optimization Methods in Finance
Fall 2010
Sheet 3

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 3.1 (*)
Consider the optimization problem

\[
\begin{align*}
\min \quad & x^2 + 1 \\
\text{subject to} \quad & (x-2)(x-4) \leq 0 \\
& x \in \mathbb{R}
\end{align*}
\]

i) Analysis of primal problem. Give the feasible set, the optimal value and the optimal solution.

ii) Lagrangian and dual function. Plot the function \(x^2 + 1\) versus \(x\). One the same plot, show the feasible set, optimal point and value, and plot the Lagrangian \(L(x, \lambda)\) versus \(x\) for a few positive values of \(\lambda\). Verify the lower bound property \(p^* \geq \inf_x L(x, \lambda)\) for \(\lambda \geq 0\). Derive and sketch the Lagrange dual function \(g\).

iii) Lagrange dual problem. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimum solution \(\lambda^*\). Does strong duality hold?

Solution:

1. One has \((x-2)(x-4) \leq 0 \iff 2 \leq x \leq 4\). The optimum solution is \(x^* = 2\) (since \(x^2 + 1\) is monotone increasing for \(x > 0\)) with value \(p^* = 2^2 + 1 = 5\).

2. One has

\[
L(x, \lambda) = (x^2 + 1) + \lambda (x-2)(x-4) = (\lambda + 1)x^2 - 6\lambda x + (1 + 8\lambda)
\]

Note that always \(g(\lambda) \leq L(\lambda, 2) = (\lambda + 1) \cdot 2^2 - 12\lambda + (1 + 8\lambda) = 0 \cdot \lambda + 5\). The quadratic function \(L(\lambda, x)\) is minimal, if \((\lambda + 1)x^2 - 6\lambda x + (1 + 8\lambda)\)' = 2(\lambda + 1)x - 6\lambda = 0 \iff x = \frac{3\lambda}{\lambda + 1}\) hence

\[
g(\lambda) = (\lambda + 1) \left(\frac{3\lambda}{\lambda + 1}\right)^2 - 6\lambda \frac{3\lambda}{\lambda + 1} + (1 + 8\lambda) = \frac{9\lambda^2 - 18\lambda^2 + \lambda + 1 + 8\lambda^2 + 8\lambda}{\lambda + 1} = \frac{-\lambda^2 + 9\lambda + 1}{\lambda + 1} +
\]
Exercise 3.2 (*)
In this exercise, we want to show an example of a convex program, where strong duality fails. Consider the optimization problem

$$
\min \ e^{-x} \\
\frac{x^2}{y} \leq 0 \\
(x,y) \in D
$$

with $D := \{ (x,y) \in \mathbb{R}^2 \mid y > 0 \}$.

i) Verify that this is a convex optimization problem. Find the optimal value.

ii) Give the Lagrange dual problem, and find the optimal solution $\lambda^*$ and optimum value $d^*$ of the dual program. What is the optimal duality gap?

iii) Does Slater’s condition hold for this problem?

Solution:

i) The function $e^{-x}$ is convex, since $(e^{-x})'' = (-e^{-x})' = e^{-x} > 0$ for all $x \in \mathbb{R}$. $D$ is clearly convex. Furthermore, consider $(x,y), (x',y') \in D$ with $x^2/y \leq 0, x'^2/y' \leq 0$ and $0 \leq \lambda \leq 1$. Then $\frac{(\lambda x + (1-\lambda)x')^2}{\lambda y + (1-\lambda)y'} \geq 0$ (nominator and denominator are non-negative).

ii) We have

$$
L(\lambda, (x,y)) = e^{-x} + \lambda \frac{x^2}{y}
$$

Then

$$
g(\lambda) = \inf_{(x,y) : y > 0} \left( e^{-x} + \lambda \frac{x^2}{y} \right) = 0
$$
for every $\lambda \geq 0$ (more precisely, we can choose $(x,y) = (\mu, \mu^3)$, then
\[
\lim_{\mu \to -\infty} L(\lambda, (\mu, \frac{1}{\mu^3})) = \lim_{\mu \to -\infty} (e^{-\mu} + \lambda : \frac{\mu^2}{\mu^3}) = 0
\]
Hence $d^* = 0$, while $p^* = \min_{(x,y) : x \geq 0, y > 0} e^{-x} = 1$. The duality gap is $p^* - d^* = 1$.

iii) If $(x,y) \in D$ is feasible, then $x^2 \leq 0$ hence $x = 0$. In other words, the feasible region is $\{(0,y) \mid y > 0\}$, which does not contain a strictly feasible point. Hence Slater’s condition is not satisfied.

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**Exercise 3.3 (**) 

In this exercise, we want to argue, why the RWMA (which can minimize convex functions over the simplex $\Sigma^m := \text{conv}\{e_1, \ldots, e_m\} = \{\lambda \in \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0\}$) can also be used to optimize over general polytopes. Here, we are motivated, since the minimum variance portfolio problem is a convex optimization problem over the domain $\{x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, \sum_{i=1}^N z_i x_i \geq r, x \geq 0\}$ which is $\Sigma^N$ intersected with the half-space $\sum_{i=1}^N \theta_i x_i \geq r$.

Let $v_1, \ldots, v_m \in \mathbb{R}^m$ and let $Q := \text{conv}\{v_1, \ldots, v_m\} := \sum_{i=1}^m \lambda_i v_i \mid \sum_{i=1}^m \lambda_i = 1, \lambda_1, \ldots, \lambda_m \geq 0\}$. Define $q : \Sigma^m \to Q$ with $q(\lambda) = \sum_{i=1}^m \lambda_i v_i$. Let $f : Q \to \mathbb{R}$ be a convex function. Show that

i) The function $g : \Sigma^m \to \mathbb{R}$ with $g(\lambda) = f(q(\lambda))$ is convex.

ii) One has
\[
\min_{\lambda \in \Sigma^m} g(\lambda) = \min_{x \in Q} f(x)
\]

iii) Describe $\Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \theta_i x_i \geq r\}$ as the convex hull of at most $N^2 + N$ points and conclude that the RWMA can be used to solve the portfolio optimization problem $\min\{x^T Q x \mid x \in \Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \theta_i x_i \geq r\}\}$.

**Solution:**

1. Let $x, y \in \Sigma^m$ and $0 \leq \theta \leq 1$, then
\[
g(\theta x + (1-\theta) y) = f(\sum_{i=1}^m v_i (\theta x_i + (1-\theta) y_i)) \leq f_{\text{convex}}(\sum_{i=1}^m x_i v_i + (1-\theta) f(\sum_{i=1}^m y_i v_i) \leq \theta g(x) + (1-\theta) g(y)
\]

2. We claim that $q$ is surjective. To see this take a $x \in Q$. By definition, there are $\lambda_1, \ldots, \lambda_m \geq 0, \sum_{i=1}^m \lambda_i = 1$ with $x = \sum_{i=1}^m v_i \lambda_i$. Then $q(\lambda) = \sum_{i=1}^m \lambda_i v_i = x$. Hence for any $x^* \in Q$ attaining $\min_{x \in Q} f(x)$ there is a $\lambda^* \in \Sigma^m$ with $q(\lambda^*) = x^*$ and hence $g(\lambda^*) = f(q(\lambda^*)) = f(x^*)$.

3. We create a set $V$ of vertices as follows: If $r_i \geq r \Rightarrow e_i \in V$. For pairs $i, j$ with $r_i \geq r$ and $r_j < r$, we add $(0, \ldots, x_i, \ldots, x_j, \ldots, 0) \in V$ with $x_i x_j = 1$ and $x_i r_j + x_j r_i = r$. Let $Q := \Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \theta_i x_i \geq r\}$. We claim that $Q = \text{conv}(V)$. Obviously $V \subseteq Q$ and hence $\text{conv}(V) \subseteq Q$. Next, suppose that $Q \setminus \text{conv}(V)$ is non-empty. Take an extreme-point $x^* \in Q \setminus \text{conv}(V)$ (there must be extreme points, since $Q$ is convex, closed and bounded). Write the following system
\[
\begin{align*}
\sum_{i=1}^N x_i & \geq 1 & (1) \\
\sum_{i=1}^N -x_i & \geq -1 & (2) \\
\sum_{i=1}^N \theta_i x_i & \geq r & (3) \\
x_i & \geq 0 & \forall i = 1, \ldots, N & (4)
\end{align*}
\]
as $Ax \geq b$. Let $A'x \leq b'$ be a maximal full-rank subsystem such that $A'x' = b'$. If rank($A'$) < $N$ then there is a vector $u \in \ker(A')$ and $x^* + \lambda u \in Q/\text{conv}(V)$ for all $-\epsilon \leq \lambda \leq \epsilon$ for a small enough $\epsilon > 0$. Then $x^*$ would not be an extreme point. Hence rank($A'$) = $N$. We have $\sum_{i=1}^{N} x_i = 1$ in the system, hence there are exactly 2 constraints from (3), (4) that are not in the system.

Case A: (3) and one constraint $j$ from (4) are not in $A'x = b'$. Then $\sum_{i=1}^{N} x_i = 1$, $x_i^* = 0 \forall i \neq j \Rightarrow x = (0, \ldots, 0, 1, 0, \ldots, 0) = e_j \in V$.

Case B: Constraints $j, j'$ from (4) are not in $A'x = b'$. Then $\sum_{i=1}^{N} x_i = 1$, $\sum_{i=1}^{N} \bar{r}_i x_i = r$, $x_i = 0 \forall i \notin \{j, j'\}$. Then

$$x^* = \left(0, \ldots, \frac{r-r_j}{r_j-r_{j'}}, \ldots, \frac{r-r_{j'}}{r_{j'}-r_j}, \ldots, 0\right) \in V$$

Hence $\text{conv}(V) = Q$.

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Exercise 3.4 (*)

Let $D \subseteq \mathbb{R}^n$ be a convex set and $f_0, \ldots, f_m : D \rightarrow \mathbb{R}$ be convex functions. Show that the set

$$A = \{(u, t) \in \mathbb{R}^m \times \mathbb{R} \mid \exists x \in D : f_i(x) \leq u_i, f_0(x) \leq t\}$$

is convex.

Solution:

Let $0 \leq \theta \leq 1$ and $(u, t), (u', t') \in A$. Then there are $x, x' \in D$ with $f_i(x) \leq u$, $f_i(x') \leq u'$, $f_0(x) \leq t$, $f_0(x') \leq t'$. Choose $x'' := \theta x + (1-\theta)x'$. Then $f_0(x'') \leq \theta f_0(x) + (1-\theta)f_0(x') \leq \theta t + (1-\theta)t'$ since $f_0$ is convex. $f_i(x'') \leq \theta f_i(x) + (1-\theta)f_i(x') \leq \theta u_i + (1-\theta)u'$ (using that $f_i$ is convex). Then $(u'', t'') := (\theta u + (1-\theta)u', \theta t + (1-\theta)t')$ lies in $A$.

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Exercise 3.5 (*)

Let $f : D \rightarrow \mathbb{R}$ be a convex function for some convex domain $D \subseteq \mathbb{R}^n$. Show that

i) The function $f(x)^2$ is convex, given that $f(x) \geq 0$ for all $x \in D$.

ii) $f(Ax + b)$ is convex for any $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Conclude that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $f(x) = x^T \cdot Q \cdot x$ and $Q \in \mathbb{R}^{n \times n}, Q \succeq 0$ is convex.

Solution:
1. Let \( x, y \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \). We have to show: 
\[
f(\lambda x + (1 - \lambda)y)^2 \leq \lambda f(x)^2 + (1 - \lambda)f(y)^2.
\]

\[
\begin{align*}
\lambda f(x)^2 + (1 - \lambda)f(y)^2 - \left( \lambda f(x) + (1 - \lambda)f(y) \right)^2 \\
\geq f(\lambda x + (1 - \lambda)y)^2 - \lambda f(x) + (1 - \lambda)f(y)
\end{align*}
\]

\[
= \lambda f(x)^2 + (1 - \lambda)f(y)^2 - \lambda^2 f(x)^2 - 2\lambda (1 - \lambda)f(x)f(y) - (1 - \lambda)^2 f(y)^2
\]

\[
= f(x)^2 \lambda (1 - \lambda) - 2\lambda (1 - \lambda)f(x)f(y) + f(y)^2 (1 - \lambda) (1 - (1 - \lambda))
\]

\[
= \lambda (1 - \lambda) \cdot (f(x)^2 - 2f(x)f(y) + f(y)^2)
\]

\[
= \lambda (1 - \lambda) (f(x) - f(y))^2
\]

\[
\geq 0
\]

In the first step, we need that \( f \) is convex, hence \( a := f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) =: b \) and that \(-a^2 \geq -b^2\) if \( 0 \leq a \leq b \).

2. Let \( x, y \in \mathbb{R}^m \) and \( \lambda \in [0, 1] \), then

\[
f(A(\lambda \cdot x + (1 - \lambda) \cdot y) + b) \quad = \quad f(\lambda \cdot (Ax + b) + (1 - \lambda) \cdot (Ay + b))
\]

\[
\begin{align*}
f(\lambda f(x) + (1 - \lambda)f(y)) & \quad \leq \quad \lambda f(Ax + b) + (1 - \lambda)f(Ay + b)
\end{align*}
\]

We conclude that: \( \|x\| \) convex \( \Rightarrow \|Ax\| \) convex (and non-negative) \( \Rightarrow \|Ax\|^2 = x^T Q x \) convex.