Prof. Friedrich Eisenbrand | Question session: 27.10.10
Location: MA A3 31 | Discussion: 03.11.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 3

Note: This is just <u>one</u> way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 3.1 (*)
Consider the optimization problem

$$\begin{array}{rcl} \min x^2+1 & & \\ (x-2)(x-4) & \leq & 0 \\ x & \in & \mathbb{R} \end{array}$$

i) *Analysis of primal problem.* Give the feasible set, the optimal value and the optimal solution.

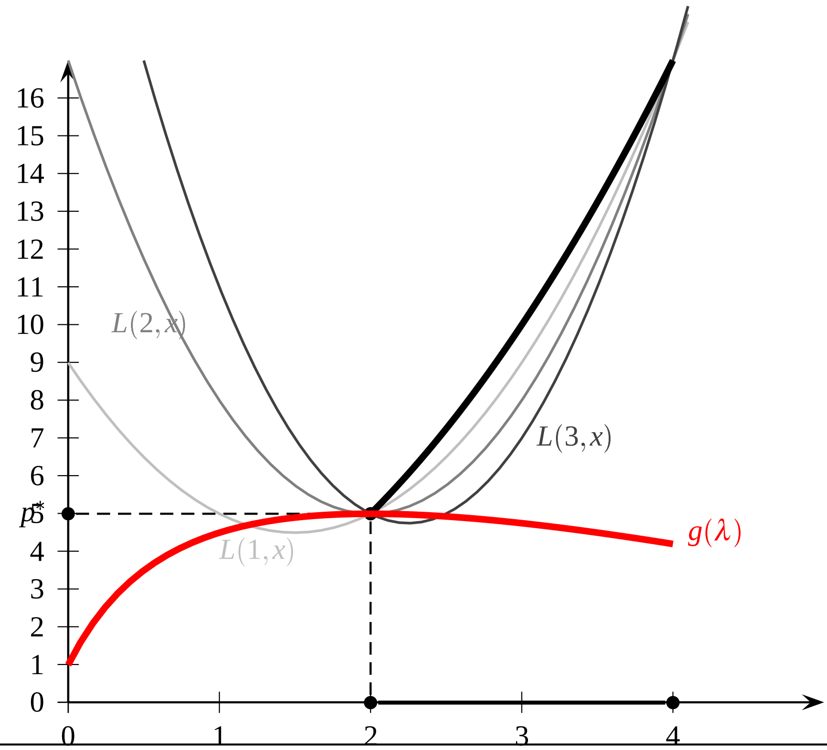
ii) *Lagrangian and dual function.* Plot the function x^2+1 versus x. One the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

iii) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimum solution λ^*. Does strong duality hold?

Solution:

1. One has $(x-2)(x-4) \leq 0 \Leftrightarrow 2 \leq x \leq 4$. The optimum solution is $x^* = 2$ (since x^2+1 is monotone increasing for $x > 0$) with value $p^* = 2^2+1 = 5$.

2. One has
$$L(x,\lambda) = (x^2+1)+\lambda(x-2)(x-4) = (\lambda+1)x^2 - 6\lambda x + (1+8\lambda)$$
Note that always $g(\lambda) \leq L(\lambda,2) = (\lambda+1)\cdot 2^2 - 12\lambda + (1+8\lambda) = 0\cdot\lambda + 5$. The quadratic function $L(\lambda,x)$ is minimal, if $((\lambda+1)x^2 - 6\lambda x + (1+8\lambda))' = 2(\lambda+1)x - 6\lambda = 0 \Rightarrow x = \frac{3\lambda}{\lambda+1}$ hence
$$g(\lambda) = (\lambda+1)\left(\frac{3\lambda}{\lambda+1}\right)^2 - 6\lambda\frac{3\lambda}{\lambda+1} + (1+8\lambda) = \frac{9\lambda^2 - 18\lambda^2 + \lambda + 1 + 8\lambda^2 + 8\lambda}{\lambda+1} = \frac{-\lambda^2+9\lambda+1}{\lambda+1} +$$



Exercise 3.2 (*)
In this exercise, we want to show an example of a convex program, where strong duality fails. Consider the optimization problem

$$\begin{aligned} \min\ & e^{-x} \\ x^2/y &\leq 0 \\ (x,y) &\in D \end{aligned}$$

with $D := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

i) Verify that this is a convex optimization problem. Find the optimal value.

ii) Give the Lagrange dual problem, and find the optimal solution λ^* and optimum value d^* of the dual program. What is the optimal duality gap?

iii) Does Slater's condition hold for this problem?

Solution:

i) The function e^{-x} is convex, since $(e^{-x})'' = (-e^{-x})' = e^{-x} > 0$ for all $x \in \mathbb{R}$. D is clearly convex. Furthermore, consider $(x,y),(x',y') \in D$ with $x^2/y \leq 0, {x'}^2/y' \leq 0$ and $0 \leq \lambda \leq 1$. Then $\frac{(\lambda x+(1-\lambda)x')^2}{\lambda y+(1-\lambda)y'} \geq 0$ (nominator and denominator are non-negative).

ii) We have

$$L(\lambda,(x,y)) = e^{-x} + \lambda \frac{x^2}{y}$$

Then

$$g(\lambda) = \inf_{(x,y):y>0} \underbrace{e^{-x}}_{\to 0, \text{for } x\to\infty} + \lambda \underbrace{\frac{x^2}{y}}_{\to 0, \text{ for } y\to\infty} = 0$$

for *every* $\lambda \geq 0$ (more precisely, we can choose $(x,y) = (\mu, \mu^3)$, then

$$\lim_{\mu\to\infty} L(\lambda, (\mu, \frac{1}{\mu^3})) = \lim_{\mu\to\infty} (e^{-\mu} + \lambda \cdot \frac{\mu^2}{\mu^3}) = 0$$

Hence $d^* = 0$, while $p^* = \min_{(x,y):x=0,y>0} e^{-x} = 1$. The duality gap is $p^* - d^* = 1$.

iii) If $(x,y) \in D$ is feasible, then $x^2 \leq 0$ hence $x = 0$. In other words, the feasible reagion is $\{(0,y) \mid y > 0\}$, which does not contain a strictly feasible point. Hence Slater's condition is not satisfied.

Exercise 3.3 (*)

In this exercise, we want to argue, why the RWMA (which can minimize convex functions over the simplex $\Sigma^m := \text{conv}\{e_1, \dots, e_m\} = \{\lambda \in \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0\}$) can also be used to optimize over general polytopes. Here, we are motivated, since the minimum variance portfolio problem is a convex optimization problem over the domain $\{x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, \sum_{i=1}^N \bar{r}_i x_i \geq r, x \geq 0\}$ which is Σ^N intersected with the half-space $\sum_{i=1}^N \bar{r}_i x_i \geq r$.

Let $v_1, \dots, v_m \in \mathbb{R}^n$ and let $Q := \text{conv}\{v_1, \dots, v_m\} := \{\sum_{i=1}^m \lambda_i v_i \mid \sum_{i=1}^m \lambda_i = 1, \lambda_1, \dots, \lambda_m \geq 0\}$. Define $q : \Sigma^m \to Q$ with $q(\lambda) = \sum_{i=1}^m \lambda_i x_i$. Let $f : Q \to \mathbb{R}$ be a convex function. Show that

i) The function $g : \Sigma^m \to \mathbb{R}$ with $g(\lambda) = f(q(\lambda))$ is convex.

ii) One has
$$\min_{\lambda\in\Sigma^m} g(\lambda) = \min_{x\in Q} f(x)$$

iii) Describe $\Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \bar{r}_i x_i \geq r\}$ as the convex hull of at most $N^2 + N$ points and conclude that the RWMA can be used to solve the portfolio optimization problem $\min\{x^T Q x \mid x \in \Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \bar{r}_i x_i \geq r\}\}$.

Solution:

1. Let $x, y \in \Sigma^m$ and $0 \leq \theta \leq 1$, then
$$g(\theta x + (1-\theta)y) = f(\sum_{i=1}^m v_i(\theta x_i + (1-\theta)y_i)) \overset{f \text{ convex}}{\leq} \theta f(\sum_{i=1}^m x_i v_i) + (1-\theta) f(\sum_{i=1}^m y_i v_i) \leq \theta g(x) + (1-\theta) g(y)$$

2. We claim that q is surjective. To see this take a $x \in Q$. By definition, there are $\lambda_1, \dots, \lambda_m \geq 0, \sum_{i=1}^m \lambda_i = 1$ with $x = \sum_{i=1}^m v_i \lambda_i$. Then $q(\lambda) = \sum_{i=1}^m \lambda_i v_i = x$. Hence for any $x^* \in Q$ attaining $\min_{x\in Q} f(x)$ there is a $\lambda^* \in \Sigma^m$ with $q(\lambda^*) = x^*$ and hence $g(\lambda^*) = f(q(\lambda^*)) = f(x^*)$.

3. We create a set V of vertices as follows: If $r_i \geq r \Rightarrow e_i \in V$. For pairs i, j with $r_i \geq r$ and $r_j < r$, we add $(0, \dots, x_i, \dots, x_j, \dots, 0) \in V$ with $x_i + x_j = 1$ and $x_i r_i + x_j r_j = r$. Let $Q := \Sigma^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \bar{r}_i x_i \geq r\}$. We claim that $Q = \text{conv}(V)$. Obviously $V \subseteq Q$ and hence $\text{conv}(V) \subseteq Q$. Next, suppose that $Q \backslash \text{conv}(V)$ is non-empty. Take an extreme-point x^* of $Q \backslash \text{conv}(V)$ (there must be extreme points, since Q is convex, closed and bounded). Write the following system

$$\begin{array}{rcll} \sum_{i=1}^N x_i & \geq & 1 & \quad (1) \\ \sum_{i=1}^N -x_i & \geq & -1 & \quad (2) \\ \sum_{i=1}^N \bar{r}_i x_i & \geq & r & \quad (3) \\ x_i & \geq & 0 \quad \forall i = 1, \dots, N & \quad (4) \end{array}$$

as $Ax \geq b$. Let $A'x \leq b'$ be a maximal full-rank subsystem such that $A'x^* = b'$. If $\text{rank}(A') < N$ then there is a vector $u \in \ker(A')$ and $x^* + \lambda u \in Q/\text{conv}(V)$ for all $-\varepsilon \leq \lambda \leq \varepsilon$ for a small enough $\varepsilon > 0$. Then x^* would not be an extreme point. Hence $\text{rank}(A') = N$. We have $\sum_{i=1}^N x_i = 1$ in the system, hence there are exactly 2 constraints from (3), (4) that are not in the system.

Case A: (3) and one constraint j from (4) are not in $A'x^* = b'$. Then $\sum_{i=1}^N x_i^* = 1,\ x_i^* = 0\ \forall i \neq j \Rightarrow x = (0,\dots,0,1,0,\dots,0) = e_j \in V$.

Case B: Constraints j, j' from (4) are not in $A'x^* = b'$. Then $\sum_{i=1}^N x_i^* = 1,\ \sum_{i=1}^N \overline{r}_i x_i = r,\ x_i = 0\ \forall i \notin \{j, j'\}$. Then

$$x^* = \Big(0,\dots,\underbrace{\frac{r - r_{j'}}{r_j - r_{j'}}}_{=x_j^*},\dots,\underbrace{\frac{r - r_j}{r_{j'} - r_j}}_{=x_{j'}^*},\dots,0\Big) \in V$$

Hence $\text{conv}(V) = Q$.

Exercise 3.4 (*)

Let $D \subseteq \mathbb{R}^n$ be a convex set and $f_0,\dots,f_m : D \to \mathbb{R}$ be convex functions. Show that the set

$$A = \{(u,t) \in \mathbb{R}^m \times \mathbb{R} \mid \exists x \in D : f_i(x) \leq u_i,\ f_0(x) \leq t\}$$

is convex.

Solution:

Let $0 \leq \theta \leq 1$ and $(u,t),(u',t') \in A$. Then there are $x, x' \in D$ with $f_i(x) \leq u,\ f_i(x') \leq u_i',\ f_0(x) \leq t,\ f_0(x') \leq t'$. Choose $x'' := \theta x + (1-\theta)x'$. Then $f_0(x'') \leq \theta f_0(x) + (1-\theta) f_0(x') \leq \theta t + (1-\theta)t' =: t''$ since f_0 is convex. $f_i(x'') \leq \theta f_i(x) + (1-\theta) f_i(x') \leq \theta u_i + (1-\theta)u_i'$ (using that f_i is convex). Then $(u'',t'') := (\theta u + (1-\theta)u', \theta t + (1-\theta)t')$ lies in A.

Exercise 3.5 (*)

Let $f : D \longrightarrow \mathbb{R}$ be a convex function for some convex domain $D \subseteq \mathbb{R}^n$. Show that

i) The function $f(x)^2$ is convex, given that $f(x) \geq 0$ for all $x \in D$.

ii) $f(Ax+b)$ is convex for any $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

Conclude that a function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^T \cdot Q \cdot x$ and $Q \in \mathbb{R}^{n\times n}$, $Q \succeq 0$ is convex.

Solution:

1. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. We have to show: $f(\lambda x + (1-\lambda)y)^2 \le \lambda f(x)^2 + (1-\lambda) f(y)^2$.

$$\begin{aligned}
& \lambda f(x)^2 + (1-\lambda) f(y)^2 - (\underbrace{f(\lambda x + (1-\lambda)y)}_{\le \lambda f(x) + (1-\lambda) f(y)})^2 \\
\overset{f \text{ convex}}{\ge} \quad & \lambda f(x)^2 + (1-\lambda) f(y)^2 - (\lambda f(x) + (1-\lambda) f(y))^2 \\
= \quad & \lambda f(x)^2 + (1-\lambda) f(y)^2 - \lambda^2 f(x)^2 - 2\lambda(1-\lambda) f(x) f(y) - (1-\lambda)^2 f(y)^2 \\
= \quad & f(x)^2 \lambda (1-\lambda) - 2\lambda(1-\lambda) f(x) f(y) + f(y)^2 (1-\lambda) \underbrace{(1-(1-\lambda))}_{=\lambda} \\
= \quad & \lambda(1-\lambda) \cdot \left(f(x)^2 - 2 f(x) f(y) + f(y)^2\right) \\
= \quad & \lambda(1-\lambda) (f(x) - f(y))^2 \\
\ge \quad & 0
\end{aligned}$$

In the first step, we need that f is convex, hence $a := f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) =: b$ and that $-a^2 \ge -b^2$ if $0 \le a \le b$.

2. Let $x, y \in \mathbb{R}^m$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
f(A(\lambda \cdot x + (1-\lambda) \cdot y) + b) \quad & = \quad f(\lambda \cdot (Ax+b) + (1-\lambda) \cdot (Ay+b)) \\
& \overset{f \text{ convex}}{\le} \quad \lambda \cdot f(Ax+b) + (1-\lambda) \cdot f(Ay+b)
\end{aligned}$$

We conclude that: $\|x\|$ convex $\Rightarrow$ $\|Ax\|$ convex (and non-negative) $\Rightarrow$ $\|Ax\|^2 = x^T Q x$ convex.