Exercises

**Optimization Methods in Finance**

Fall 2010

Sheet 3

Exercises marked with (*) qualify for one bonus point, if correctly presented in the discussion session

**Exercise 3.1 (*)**
Consider the optimization problem

\[
\begin{align*}
\min & \quad x^2 + 1 \\
\text{s.t.} & \quad (x - 2)(x - 4) \leq 0 \\
& \quad x \in \mathbb{R}
\end{align*}
\]

i) *Analysis of primal problem.* Give the feasible set, the optimal value and the optimal solution.

ii) *Lagrangian and dual function.* Plot the function \(x^2 + 1\) versus \(x\). One the same plot, show the feasible set, optimal point and value, and plot the Lagrangian \(L(x, \lambda)\) versus \(x\) for a few positive values of \(\lambda\). Verify the lower bound property (\(p^* \geq \inf \lambda L(x, \lambda)\) for \(\lambda \geq 0\)). Derive and sketch the Lagrange dual function \(g\).

iii) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimum solution \(\lambda^*\). Does strong duality hold?

**Exercise 3.2 (*)**
In this exercise, we want to show an example of a convex program, where strong duality fails. Consider the optimization problem

\[
\begin{align*}
\min & \quad e^{-x} \\
\text{s.t.} & \quad \frac{x^2}{y} \leq 0 \\
& \quad (x,y) \in D
\end{align*}
\]

with \(D := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}\).

i) Verify that this is a convex optimization problem. Find the optimal value.

ii) Give the Lagrange dual problem, and find the optimal solution \(\lambda^*\) and optimum value \(d^*\) of the dual program. What is the optimal duality gap?

iii) Does Slater’s condition hold for this problem?
Exercise 3.3 (*)
In this exercise, we want to argue, why the RWMA (which can minimize convex functions over the simplex $\Sigma^m := \text{conv}\{e_1, \ldots, e_m\} = \{\lambda \in \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0\}$) can also be used to optimize over general polytopes. Here, we are motivated, since the minimum variance portfolio problem is a convex optimization problem over the domain $\{x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, \sum_{i=1}^N r_i x_i \geq r, x \geq 0\}$ which is $\sum^N$ intersected with the half-space $\sum_{i=1}^N 7_i x_i \geq r$.

Let $v_1, \ldots, v_m \in \mathbb{R}^n$ and let $Q := \text{conv}\{v_1, \ldots, v_m\} := \{\sum_{i=1}^m \lambda_i v_i \mid \sum_{i=1}^m \lambda_i = 1, \lambda_1, \ldots, \lambda_m \geq 0\}$. Define $q : \Sigma^m \to Q$ with $q(\lambda) = \sum_{i=1}^m \lambda_i v_i$. Let $f : Q \to \mathbb{R}$ be a convex function. Show that

i) The function $g : \Sigma^m \to \mathbb{R}$ with $g(\lambda) = f(q(\lambda))$ is convex.

ii) One has
$$\min_{\lambda \in \Sigma^m} g(\lambda) = \min_{x \in Q} f(x)$$

iii) Describe $\sum^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N 7_i x_i \geq r\} \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N 7_i x_i \geq r\}$ as the convex hull of at most $N^2 + N$ points and conclude that the RWMA can be used to solve the portfolio optimization problem $\min\{x^T Q x \mid x \in \sum^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N 7_i x_i \geq r\}\}$.

Exercise 3.4 (*)
Let $D \subseteq \mathbb{R}^n$ be a convex set and $f_0, \ldots, f_m : D \to \mathbb{R}$ be convex functions. Show that the set
$$A = \{(u, t) \in \mathbb{R}^m \times \mathbb{R} \mid \exists x \in D : f_i(x) \leq u_i, f_0(x) \leq t\}$$
is convex.

Exercise 3.5 (*)
Let $f : D \rightarrow \mathbb{R}$ be a convex function for some convex domain $D \subseteq \mathbb{R}^n$. Show that

i) The function $f(x)^2$ is convex, given that $f(x) \geq 0$ for all $x \in D$.

ii) $f(Ax + b)$ is convex for any $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Conclude that a function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^T \cdot Q \cdot x$ and $Q \in \mathbb{R}^{n \times n}, Q \geq 0$ is convex.