Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 13.10.10
Discussion: 20.10.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 2

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 2.1 (*)
Consider again the simple setting, where we have N experts that (over a time horizon of T units) predict a binary event ($y_j^t \in \{0,1\}$) and a forecaster tries to predict the events so that he is not making significantly more mistakes than the best of the experts. Consider the following strategies:

- *Strategy 1:* The forecaster chooses at any time t the prediction $\hat{p}_t$ of the expert j who made the least number of mistakes so far (i.e. $\hat{p}_t = y_j^t$ where $j = \text{argmin}\{m_j\}$ and $m_j = |\{t' < t \mid y_j^{t'} \neq z_{t'}\}|$ is the number of mistakes, which were made by expert j in time $1,\dots,t-1$). If several experts have the same minimal number of mistakes, we choose that one with a smaller index j.

- *Strategy 2:* The forecaster chooses the prediction of expert j with probability

$$\frac{t-m_j}{\sum_{j'=1}^{N}(t-m_{j'})}$$

(i.e. proportional to the number of correct predictions; say in the first iteration, we choose an expert uniformly at random).

Show that both strategies can be much worse (say for $T \gg N$ and suitable ε) than the weigthed majority experts algorithm (Algorithm 2 from the lecture).

Solution:

For strategy 1, imagine we have only 2 experts. Expert 1 predicts correct on even days and expert 2 predicts correct on odd days. After odd days, we decide for expert 2 who will be wrong next time. After even days, both expert have the same number of mistakes, hence we decide for expert 1 (which again will be wrong). Hence the forcaster will *always* be wrong, while both experts are correct half of the time.

For strategy 2, we imagine to have an expert 1 that is always correct. The other experts are all correct exactly on odd days, on even days they are all wrong. Then the probability to choose expert 1 tends to $\frac{t}{t+(N-1)\cdot t/2} \leq \frac{2}{N+1}$. Hence, on even days, we will be wrong with probability essentially $1-\frac{2}{N+1}$. In other words, while there is a perfect expert, the forecaster makes wrong predictions essentially half of the times.

Exercise 2.2 (*)
Consider again the setting with N experts and loss vectors $\ell^t \in [0,1]^N$. Let T be the number of iterations, $\hat{L}$ be the forecasters loss and L_j be the loss of expert j. In the lecture we saw the bound

$$E[\hat{L}] \leq \frac{\ln(N)}{\varepsilon} + (1+\varepsilon)L^j.$$

Observe that this just bounds the *average loss* of the forecaster. Can you give a concentration bound statement of the form $\Pr[\hat{L} > (1+\ldots)\cdot L^j + \ldots] \leq \ldots$. Here the following theorem (a.k.a. *Azuma's Inequality*) might be helpful (which you may use without proving it):

> *Let* $0 = X_0, X_1, \ldots, X_n$ *be a sequence of random variables with increment* $Y_i := X_i - X_{i-1}$. *Here* $Y_i := Y_i(X_0, \ldots, X_{i-1})$ *might arbitrarily depend on* $X_0, \ldots, X_{i-1}$, *but always* $|Y_i| \leq 1$ *and* $E[Y_i] = 0$. *For* $\lambda \geq 0$ *one has* $\Pr[X_n \geq \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$.

Solution:
Let $Y_t := \hat{\ell}^t - \hat{p}^t \ell^t$ be the deviation from the average loss in step t and $X_t = \sum_{t'=0}^{t} Y_t$ be the cumulated loss. Note that $E[Y_t] = E[\hat{\ell}^t] - \hat{p}^t \ell^t = 0$ and $|Y_t| \leq 1$ hence $X_0, \ldots, X_T$ is a martingale. Hence

$$\Pr[\underbrace{X_T}_{=\hat{L}-E[\hat{L}]} \geq \lambda\sqrt{T}] \leq e^{-\lambda^2/2}$$

which implies that

$$\Pr[\hat{L} \geq \underbrace{E[\hat{L}]}_{\leq \frac{\ln(N)}{\varepsilon}+(1+\varepsilon)L^j} + \lambda\sqrt{T}] \leq e^{-\lambda^2/2}$$

hence

$$\Pr\left[\hat{L} \geq \frac{\ln(N)}{\varepsilon} + \lambda\sqrt{T} + (1+\varepsilon)L^j\right] \leq e^{-\lambda^2/2}$$

Exercise 2.3 (*)
Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, if $dom(f)$ is a convex set and for all $x, y \in dom(f)$ and $0 \leq \lambda \leq 1$ one has $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. Prove that if $f_1, \ldots, f_n : K \to \mathbb{R}$ are convex, $\lambda_1, \ldots, \lambda_n \geq 0$, then also $\sum_{i=1}^n \lambda_i f_i(x)$ is convex.

Solution:
Let $\lambda \in [0,1]$ and $f(x) := \sum_{i=1}^n \lambda_i f_i(x)$. Then for any $x, y \in K$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sum_{i=1}^n \lambda_i f_i(\lambda x + (1-\lambda)y) \overset{f_i \text{ convex}}{\leq} \sum_{i=1}^n \lambda_i(\lambda f_i(x) + (1-\lambda) f_i(y)) \\ &= \lambda\,\underbrace{(\sum_{i=1}^n \lambda_i f_i(x))}_{=f(x)} + (1-\lambda)\underbrace{\sum_{i=1}^n \lambda_i f_i(y)}_{=f(y)} = \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

Exercise 2.4 (*)
Let $y \in \mathbb{R}^n$ be a vector with $y_i > 0$ for all $i = 1, \dots, n$ and $x \in \Sigma^n$. Prove

$$\|\nabla(-\ln(y^T x))\|_\infty \le \max_{i,j} \left| \frac{y_i}{y_j} \right|$$

Note: The gradient is w.r.t. x as variable.

Solution:
Note that

$$\frac{d}{dx_i}(-\ln(y^T x)) = \frac{d}{dx_i}(-\ln(\sum_{j \neq i} y_j x_j + x_i y_i)) = -\frac{y_i}{\sum_{j \neq i} y_j x_j + x_i y_i} = -\frac{y_i}{y^T x}$$

Hence

$$\|\nabla(-\ln(y^T x))\|_\infty = \left| \frac{\max_i y_i}{y^T x} \right| \le \frac{\max_i y_i}{\underbrace{\sum_{i'=1}^n x_{i'}}_{=1} \min y_i} = \max_{i,j} \left| \frac{y_i}{y_j} \right|.$$

Exercise 2.5 (one practical bonus point)
Recall the example from the lecture

	Stock A	Stock B	Money Market
Up	2	1.5	1
Stable	1.2	1.7	1.3
Down	0.8	1.2	1.4

Implement the presented algorithm to determine an optimum row strategy. Choose $\varepsilon := 0.1$, $\delta := 0.2$ and run the algorithm for $T = 100$ iterations.

The details for the submission are as follows:

1. You can implement the algorithm in one of the programming languages C/C++/Java/Pascal/Basic/Matlab (you can choose your favourite one).

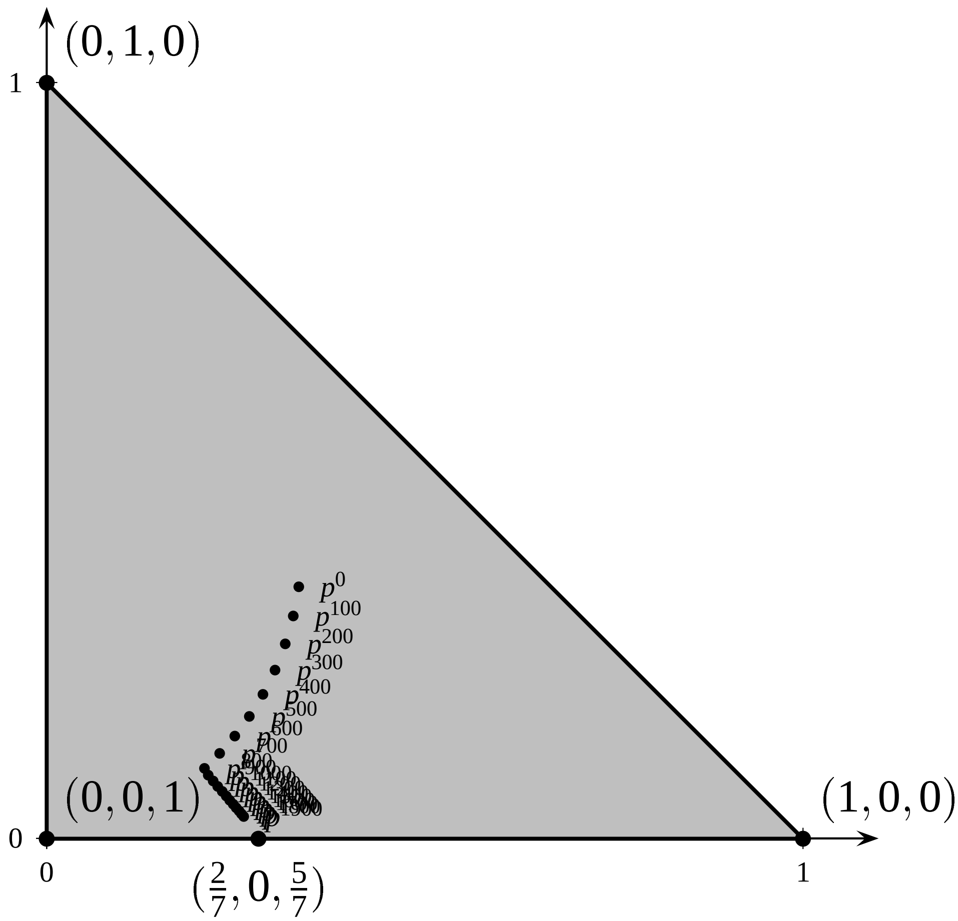
2. Your submission should contain your (compilable) code together with an output of the algorithm, which states t, w^t, p^t, j_t for all iterations $t = 0, \dots, 100$.

3. Send the files till **20.10.10** to thomas.rothvoss@epfl.ch.

4. You can work in groups up to 3 people (you need only one submission per group).

Solution:

The sequence of strategies $p^0,\ldots,p^T$ in the simplex Σ^n can be visualized as follows (we draw the strategy p^t every 100th iteration).



We can easily convince ourselves that $Q=(\frac{2}{7},0,\frac{5}{7})$ is an optimal strategy for the row-player and $(0,\frac{4}{7},\frac{3}{7})$ is an optimal column strategy since

$$\begin{aligned} & \begin{pmatrix} 2/7 & 0 & 5/7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1.5 & 1 \\ 1.2 & 1.7 & 1.3 \\ 0.8 & 1.2 & 1.4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 4/7 \\ 3/7 \end{pmatrix} \\ = & \begin{pmatrix} 2/7 & 0 & 5/7 \end{pmatrix} \cdot (9/7, 107/70, 9/7) \\ = & \begin{pmatrix} 8/7 & 9/7 & 9/7 \end{pmatrix} \cdot (0, 4/7, 3/7) \end{aligned}$$

In other words, if the row player starts playing, the (maximizing) column player has no incentive to deviate from his strategy and vice versa. The payoff is $9/7$.