Thomas Rothvoß
Location: ELD120 Discussion: 2.06.10

Exercises

Approximation Algorithms

Spring 2010

Sheet 12

Announcement: Please note that there is no lecture on May 26 (I'm on a workshop).
The next (and last) lecture/exercise is June 2.

Exercise 1

An undirected graph $G = (V,E)$ is called *k-colorable* if there is a function $\chi : V \to \{1,\ldots,k\}$ such that for any edge $(i, j) \in E$ one has $\chi(i) \neq \chi(j)$.

i) Suppose that G is 3-colorable. Show that the vector program

$$\begin{aligned} v_i \cdot v_j &\leq -\frac{1}{2} \quad \forall (i, j) \in E \\ v_i \cdot v_i &= 1 \quad \forall i = 1,\ldots,n \\ v_i &\in \mathbb{R}^n \end{aligned}$$

has a solution ($n = |V|$).

ii) Let $v_1,\ldots,v_n \in \mathbb{Q}^n$ be a solution to the above system. For a parameter $t \in \mathbb{N}$ apply the following rounding scheme:

(1) Pick random vectors $r_1,\ldots,r_t$ independently from the n-dimensional unit ball

(2) Define 2^t groups of vertices

$$\begin{aligned} R_1 &= \{i \in V \mid r_1 \cdot v_i \geq 0, r_2 \cdot v_i \geq 0,\ldots,r_t \cdot v_i \geq 0\} \\ R_1 &= \{i \in V \mid r_1 \cdot v_i < 0, r_2 \cdot v_i \geq 0,\ldots,r_t \cdot v_i \geq 0\} \\ &\vdots \\ R_{2^t} &= \{i \in V \mid r_1 \cdot v_i < 0, r_2 \cdot v_i < 0,\ldots,r_t \cdot v_i < 0\} \end{aligned}$$

(in other words, nodes i, j are in different groups if and only if v_i and v_j are separated by at least one of the t hyperplanes with normal vector $r_1,\ldots,r_t$.)

(3) Color vertices in R_c with color c.

Which upper bound on $\Pr[i \text{ and } j \text{ get the same color}]$ can you get for an edge $(i, j) \in E$?

iii) For comparison: If you choose a color for each node i just randomly (and independently) from $1,\ldots,2^t$, what is the probablity that a given pair of nodes i and j gets the same color?

iv) Suppose that the degree of G is upper bounded by Δ (i.e. $|\delta(v)| \leq \Delta \; \forall v \in V$). Give a polynomial time algorithm that, with probability at least $1/2$, needs at most $O(\log n) \cdot \Delta^{0.64}$ many colors.
Hint: Run *ii*) with $t := \log_3(2\Delta)$ for $O(\log n)$ iterations until all nodes are feasibly colored.

Exercise 2

We are given a set $S = \{x_1, \ldots, x_n\}$ of n items and a set m triplets $T \subseteq S \times S \times S$. Each triplet consists of 3 distinct items. A total ordering of S is an ordering π, $x_{\pi(1)} \prec x_{\pi(2)} \prec \ldots \prec x_{\pi(n)}$ (here $x_i \prec x_j$ means that x_i appears before x_j in the ordering). Such an order *satisfies* a triplet $(x_i, x_j, x_k) \in T$ if either $x_i \prec x_j \prec x_k$ or $x_k \prec x_j \prec x_i$ holds. The problem is to find an ordering π that maximizes the number of satisfied triplets.

i) Argue, why a random permutation satisfies $\frac{1}{3}$ of the triplets.

ii) Suppose that all triplets can be satisfied. Show that then the following quadratic program has a solution:

$$\begin{aligned} (p_i - p_j)^2 &\geq 1 \quad \forall i \neq j \\ (p_i - p_j)(p_k - p_j) &\leq 0 \quad \forall (x_i, x_j, x_k) \in T \\ p_i &\in \mathbb{R} \quad \forall i \in \{1, \ldots, n\} \end{aligned}$$

iii) Relax this quadratic program to a vector program with variables $v_1, \ldots, v_n \in \mathbb{R}^n$.

iv) Let $v_1, \ldots, v_n$ be any solution to the vector program. Select a vector $r \in \mathbb{R}^n$ uniformly at random from the unit sphere. Consider the ordering obtained by sorting $r^T v_i$. Show that a triplet $(x_i, x_j, x_k) \in T$ is satisfied with probability $\geq 1/2$ by this ordering.
Hint: Look at the angle between $v_i - v_j$ and $v_k - v_j$.