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Exercises

Approximation Algorithms

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Sheet 11

Exercise 1
Here, we want to consider NON-METRIC FACILITY LOCATION, where facilities F with open cost f_i for $i \in F$, cities C and connection cost c_{ij} are given. Differently from the variant, studied in the lecture, we do not assume anymore, that c is metric.

i) Model the problem as SET COVER problem and obtain a polynomial time $O(\log n)$-approximation ($n := |C|$) by using the greedy algorithm for SET COVER.
Hint: Even if the defined set system has exponentially many sets, under some conditions the greedy algorithm can still be made to run in polynomial time.

ii) A result of Raz and Safra (1997) says the following:

> There is a constant $c > 0$ such that, given a SET COVER instance $S_1, \ldots, S_m$ and a parameter $k \in \mathbb{N}$ it is **NP**-hard to distinguish
>
> - YES: $OPT_{\text{SetCover}} \leq k$
> - NO: $OPT_{\text{SetCover}} \geq k \cdot c \cdot \log n$
>
> Here OPT_{SetCover} denotes the smallest number of sets that are needed to cover all n elements.
> **Remark:** This result means that there is a polynomial time reduction, taking a SAT clause $\mathscr{C}$ as instance and mapping it to a SET COVER instances $\mathscr{S} = \{S_1, \ldots, S_m\}$ such that: If $\mathscr{C}$ is satisfiable, then $OPT_{\text{SetCover}}(\mathscr{S}) \leq k$ and $OPT_{\text{SetCover}}(\mathscr{S}) \geq k \cdot c \cdot \log n$ otherwise (for more details on gap reductions, I recommend Chapter 29 of Vazirani's book *Approximation Algorithms*).

Show that it is also **NP**-hard to approximate NON-METRIC FACILITY LOCATION by a factor better than $c \cdot \log n$.

Solution:

i) For any subset $C' \subseteq C$ of cities and facility $i \in C$, we define a set $S_{C',i} = C'$ of cost $c(S_{C',i}) = f_i + \sum_{j \in C'} c_{ij}$. The NON-METRIC FACILITY LOCATION is equivalent to the arising SET COVER instance. The greedy algorithm now performs as follows:

(1) $\mathscr{S}' := \emptyset$

(2) WHILE not yet all elements covered DO

(3) $price(S) := \dfrac{c(S)}{|S \setminus \bigcup_{S' \in \mathscr{S}'} S'|}$

(4) $\mathscr{S}' := \mathscr{S}' \cup \{$ set S with minimum $price(S)\}$

The algorithm gives a $O(\log n)$-approximation for SET COVER and hence also for NON-METRIC FACILITY LOCATION, where $n = |C|$ is the number of elements/cities. The algorithm covers at least one element per iteration, hence the number of iterations is at most n. Just the number of sets is exponentially large. Thus we have to argue, that the set $S_{C,i}$, minimizing the price can be found efficiently. Consider any iteration and let $\bar{C} \subseteq C$ be the not yet covered cities. Note that

$$\min_{i\in F, C'\subseteq \bar{C}} \left\{ \frac{c(S_{i,C'})}{|C'|} \right\} = \min_{i\in F, k\in\{1,\dots,\bar{C}\}} \min_{C'\subseteq \bar{C}} \left\{ \frac{f_i + \sum_{j\in C'} c_{ij}}{k} \right\}$$

We try out all possibilities for i and k.

$$\min_{C'\subseteq \bar{C}} \left\{ \frac{f_i + \sum_{j\in C'} c_{ij}}{k} \right\} = \frac{f_i}{k} + \frac{1}{k} \min_{C'\subseteq \bar{C}} \left\{ \sum_{j\in C'} c_{ij} \right\}$$

But the latter minimum is attained for the k cities that are closest to i (which can be easily obtained by sorting the cities according to their distance to i).

ii) Sei $S_1,\dots,S_m$ be the SET COVER instance on elements $1,\dots,n$. Choose the NON-METRIC FACILITY LOCATION instance with $f_i := 1$ (one facility per set) and distances

$$c_{ij} = \begin{cases} 0 & \text{if } j \in S_i \\ m & \text{otherwise} \end{cases}$$

Then $OPT_{\text{SETCOVER}} = OPT_{FL}$. Note that this cost function is in general not metric.

Exercise 2

We consider the FACILITY LOCATION problem, with given facilities F, cities C, opening cost f_i for every facility i. Assume that the cost function c_{ij} is *metric*. In this exercise, we want to show that there is no 1.46-approximation algorithm for the (metric) FACILITY LOCATION problem.

For the sake of contradiction, suppose that we have a polynomial time algorithm $\texttt{algo}(F, C, c_{ij}, f_i)$ that produces a 1.46-approximate solution $F' \subseteq F$ (note that knowing the set of open facility suffices — the cities are then automatically connected to the nearest such facility).

Let $S_1,\dots,S_m$ be a SET COVER instance (with unit cost per set) on elements $\{1,\dots,n\}$. We may assume to know the value k of sets that are contained in an optimum solution. We will now show, how to obtain a $0.999 \cdot \ln(n) + O(1)$ approximate SET COVER solution in polynomial time. This would then contradict an inapproximability result of Feige (1998) (given that **NP** is not contained in **DTIME**$(n^{O(\log\log n)})$).

We use the following SET COVER algorithm:

(1) Let $C := \{1,\dots,n\}$, $F := \{1,...,m\}$ and $c_{ij} := \begin{cases} 1 & j \in S_i \\ 3 & \text{otherwise} \end{cases}$

(2) WHILE $C \neq \emptyset$ DO

(3) Let $f_i := 0.46 \cdot \frac{|C|}{k}$ be the facility cost $\forall i \in F$

(4) $F' := \texttt{algo}(F, C, c_{ij}, f_i)$

(5) Buy the sets in F'

(6) $C' :=$ cities covered at cost 1; set $C := C \setminus C'$

(7) Return the bought sets

Perform the following analysis:

i) Consider any iteration and let APX be the cost of the FACILITY LOCATION solution F'. Show that $APX \leq 1.46^2 \cdot |C|$.

ii) Suppose the algorithm needs T iterations. For iteration $t \in \{1, \ldots, T\}$, define β_t and α_t such that $|F'| = \beta_t k$ is the number of opened facilities and $|C'| = \alpha_t |C|$ is the number of elements that are covered in this iteration. Show that $\beta_t \leq 0.999 \cdot \ln(\frac{1}{1-\alpha_t})$ holds for any $t < T$.
Hint: It is OK if your solution contains the phrase "*By a Maple/Matlab plot we see that..*".

iii) Why is $\prod_{t=1}^{T-1}(1-\alpha_t) \geq \frac{1}{n}$?

iv) Show that the algorithm needs at most $0.999 \cdot \ln(n) \cdot k$ many sets (plus $O(k)$ for the last iteration).

Solution:

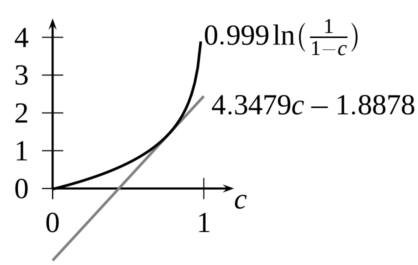
i) Let $f := f_i$. One could open the k facilities that correspond to sets in the optimum SET COVER solution and connect all clients at cost 1. This would cost in total $|C| + k \cdot f = |C| + 0.46\frac{|C|}{k} \cdot k = 1.46 \cdot |C|$. Since we assume to have a 1.46-apx algorithm, we have $APX \leq 1.46^2 \cdot |C|$.

ii) Suppose that βk centers are opened and $c|C| = |C'|$ many clients are connected at cost 1 (the others are connected at cost 3. Then this solution costs

$$\begin{aligned} f \cdot \beta \cdot k + c|C| + 3(|C| - c|C|) &= \beta 0.46|C| + c|C| + 3(|C| - c|C|) \\ &= |C| \cdot (0.46\beta + c + 3 - 3c) \\ &= |C| \cdot (0.46\beta - 2c + 3) \end{aligned}$$

On the other hand, we know that the solution costs at most $1.46^2 \cdot |C|$. Hence

$$|C| \cdot (0.46\beta - 2c + 3) \leq 1.46^2 \cdot |C| \quad \Rightarrow \quad \beta \leq 4.3479c - 1.8878 \overset{\text{Maple}}{\leq} 0.999 \ln\left(\frac{1}{1-c}\right)$$



iii) After $T-1$ iterations the number of remaining elements is precisely

$$n \cdot \prod_{t=1}^{T-1}(1-\alpha_t) \geq 1.$$

Rearranging yields the claim.

iv) The number of chosen sets in iteration $t = 1, \ldots, T-1$ is

$$\sum_{t=1}^{T-1} k\beta_t \overset{ii)}{\leq} k \sum_{t=1}^{T-1} 0.999 \cdot \ln\left(\frac{1}{1-\alpha_t}\right) = 0.999k \cdot \ln\left(\prod_{t=1}^{T-1} \frac{1}{1-\alpha_t}\right) \overset{iii)}{\leq} 0.999 \cdot \ln(n) \cdot k$$

In the last iteration we will not use more than $O(k)$ sets anyway (since $\beta \leq 4.3479c - 1.8878 \leq 3$).