Exercises

Optimization Methods in Finance

Fall 2010

Sheet 1

Exercises marked with (*) qualify for one bonus point, if correctly presented in the discussion session

**Exercise 1.1**
Let $X$ be a finite set of $N$ elements (assume that $N$ is a power of 2). At least one of these elements is *interesting*. You may ask questions of the form: *Does $H \subseteq X$ contain an interesting element?* Design an algorithm that identifies an interesting element by asking at most $\log_2(N)$ questions of this kind.

Can there exist a deterministic algorithm that identifies an interesting element by asking fewer questions in the worst case?

**Exercise 1.2**
Show the following inequalities for $0 \leq \varepsilon \leq 1/2$:

1. $(1 - \varepsilon)^x \leq (1 - \varepsilon x)$ for $x \in [0, 1]$.
2. $(1 + \varepsilon)^{-x} \leq (1 - \varepsilon x)$ for $x \in [-1, 0]$.
3. $\ln \left(\frac{1}{1-\varepsilon}\right) \leq \varepsilon + \varepsilon^2$.
4. $\ln(1 + \varepsilon) \geq \varepsilon - \varepsilon^2$.

**Exercise 1.3 (*)**
Consider the randomized weighted majority algorithm and suppose that the loss-vectors at time $t$ satisfy $\ell^t \in [0, \rho]^N$ for $t = 0, \ldots, T$. Show that the expected loss of the forecaster is bounded by

$$E[L] \leq \frac{\rho \cdot \ln N}{\varepsilon} + (1 + \varepsilon) \cdot L^j,$$

if one uses the update rule $w_j := w_j (1 - \varepsilon)^{\ell^t_j / \rho}$. As in the lecture, $L^j = \sum_{t=0}^{T} \ell^t_j$ is the loss accumulated by expert $j$.

**Exercise 1.4 (*)**
Suppose that the loss vectors satisfy $\ell^t \in [-1, 1]^N$ for $t = 0, \ldots, T$ and consider the following updating rule

$$w_j^{t+1} := \begin{cases} w_j^t (1 - \varepsilon)^{\ell^t_j} & \text{if } \ell^t_j \geq 0 \\ w_j^t (1 + \varepsilon)^{-\ell^t_j} & \text{if } \ell^t_j < 0. \end{cases}$$
Show that
\[ E[L] \leq \frac{\ln N}{\varepsilon} + (1 + \varepsilon) \cdot \sum_{t: \ell_j^t \geq 0} \ell_j^t + (1 - \varepsilon) \cdot \sum_{t: \ell_j^t < 0} \ell_j^t \]
holds. Show furthermore that this is also guaranteed if the update rule \( w_j^{t+1} := w_j(1 - \varepsilon \ell_j^t) \) is used.

**Exercise 1.5 (*)**

Suppose you have some initial belief about the quality of the experts. This belief is represented by a probability distribution on the experts \( p_j \), \( j = 1, \ldots, N \) with \( p_j > 0 \) and \( \sum_{j=1}^{N} p_j = 1 \). We modify the weighted majority algorithm by setting the initial weights \( w_j := p_j \). Show that this modification results in a guarantee
\[ E[L] \leq \frac{\ln(1/p_j)}{\varepsilon} + (1 + \varepsilon) \cdot L^j. \]

Suppose now that we have a countably infinite number of experts. Use the result above to argue that one can guarantee
\[ E[L] \leq \frac{2 \cdot \ln(j) + 10}{\varepsilon} + (1 + \varepsilon) \cdot L^j. \]
by choosing a suitable probability distribution on the experts.