Integer Points in Polyhedra

Spring 2009

Assignment Sheet 9

Exercise 1

Let C be a unimodular cone. Prove that the dual cone C^* is also unimodular.

Exercise 2

Let a_1 and a_2 be two relatively prime positive integers and let S be the set of all non-negative integral combinations of a_1 and a_2:

$$S = \{\lambda_1 a_1 + \lambda_2 a_2 : \lambda_1, \lambda_2 \in \mathbb{Z}_+\}.$$

Show that

$$\sum_{m \in S} x^m = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}.$$

Exercise 3

Let a_1, a_2 and a_3 be pairwise relatively prime positive integers and let S be the set of all non-negative integral combinations of a_1, a_2 and a_3:

$$S = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_+\}.$$

Show that

$$\sum_{m \in S} x^m = \frac{1 - x^{b_1} - x^{b_2} - x^{b_3} + x^{b_4} + x^{b_5}}{(1 - x^{a_1})(1 - x^{a_2})(1 - x^{a_3})}.$$

Exercise 4

Let $a_1, a_2, \ldots, a_n$ and $r_1, r_2, \ldots, r_n$ be positive integers and suppose that the sets $r_i + a_i\mathbb{Z}$ form a partition of $\mathbb{Z}$, i.e.,

$$\mathbb{Z} = (r_1 + a_1\mathbb{Z}) \cup (r_2 + a_2\mathbb{Z}) \cup \ldots \cup (r_n + a_n\mathbb{Z})$$

and $(r_i + a_i\mathbb{Z}) \cap (r_j + a_j\mathbb{Z}) = \emptyset$ for $i \neq j$. Prove that there are $i \neq j$ such that $a_i = a_j$.