Integer Points in Polyhedra

Spring 2009

Assignment Sheet 7

Exercise 1 (Minkowski sum)
Show that there is a unique bilinear operation $* : \mathscr{P}(\mathbb{R}^n) \times \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^n)$ such that

$$[P] * [Q] = [P + Q]$$

for any two polyhedra P and Q in $\mathbb{R}^n$.

Exercise 2 (Polarity)
Show that there is a linear transformation $\mathscr{D} : \mathscr{P}\mathbb{R}^n \to \mathscr{P}\mathbb{R}^n$ such that

$$\mathscr{D}([P]) = [P^*]$$

for any non-empty polyhedron polyhedron P in $\mathbb{R}^n$, where P^* denotes the polar of P.

Exercise 3
Let $f_1, f_2 \in \mathscr{P}(\mathbb{R}^n)$ be linear combinations of indicator functions of poyhedral cones. Prove that $\mathscr{D}(f_1 f_2) = \mathscr{D}(f_1) * \mathscr{D}(f_2)$. Here $\mathscr{D} : \mathscr{P}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)$ is the linear transformation from Exercise 2, while $* : \mathscr{P}(\mathbb{R}^n) \times \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^n)$ is the multiplication operation from Exercise 1.

Exercise 4
Let $\mathscr{P}_0(\mathbb{R}^n)$ denote the vectors space spanned by the indicators of all polyhedra containing a line. We saw that

$$[P] \equiv \sum_{v \in \mathrm{Vert}(P)} [\mathrm{cone}(P, v)] \pmod{\mathscr{P}_0(\mathbb{R}^n)}$$

holds for all simplices P in $\mathbb{R}^n$. Using this fact, show that the same is true for all polytopes in $\mathbb{R}^n$.

Exercise 5 (Gram–Brianchon theorem)
Let $P \subseteq \mathbb{R}^n$ be a polyhedron. We say that two points $x, y \in P$ are equivalent, if $\mathrm{cone}(P, x) = \mathrm{cone}(P, y)$. An equivalence class of points in P is just an open face $F \subseteq P$. For an $x \in F$, we denote $\mathrm{cone}(x)$ by $\mathrm{cone}(F)$. Prove that

$$[P] \equiv \sum_{F} (-1)^{\dim(F)} [\mathrm{cone}(P, F)],$$

where the sum is taken over all non-empty faces of P, including $F = P$.