Exercise 1 (Inclusion–exclusion)
Let $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}^n$ be sets. Prove the following inclusion–exclusion formula:

$$\left[ \bigcup_{i=1}^{n} A_i \right] = \sum_{I} (-1)^{|I|-1} \left[ \bigcap_{i \in I} A_i \right],$$

where $[A] : \mathbb{R}^n \to \mathbb{R}$ denotes the indicator function of the set $A$, $|I|$ is the cardinality of the set $I$, and the sum in the right-hand side is taken over all non-empty subsets $I$ of $\{1, 2, \ldots, n\}$.

Exercise 2 (Euler characteristic)
Show that the Euler characteristic can be extended to the space spanned by the indicators $[A]$ of closed convex sets $A \subseteq \mathbb{R}^n$ so that $\chi([A]) = 1$ if $A$ is a non-empty closed convex set.

Exercise 3 (Euler–Poincaré formula)
Let $P \subseteq \mathbb{R}^n$ be a full-dimensional polytope. Show that $\text{int}(P) \in \mathcal{P}(\mathbb{R}^n)$ and that $\chi(\text{int}(P)) = (-1)^n$. Deduce the Euler–Poincaré formula: if $P$ is an $n$-dimensional polytope, then

$$\sum_{i=1}^{n} (-1)^i f_i = 1,$$

where $f_i$ is the number of $i$-dimensional faces of $P$.

Exercise 4 (Polarity)
For a set $X \subseteq \mathbb{R}^n$, the polar $X^*$ of $X$ is the set

$$X^* := \{ z \in \mathbb{R}^n : z^T x \leq 1 \text{ for all } x \in X \}.$$

Show that if $P$ is a polyhedron in $\mathbb{R}^n$ such that $0 \in P$, then

(a) $P^*$ is a polyhedron;

(b) $P^{**} = P$;

(c) $x \in P$ if and only if $\forall z \in P^* : z^T x \leq 1$;

(d) if $P = \text{conv}(0, x_1, x_2, \ldots, x_m) + \text{cone}(y_1, y_2, \ldots, y_k)$, then

$$P^* = \{ z \in \mathbb{R}^n : z^T x_i \leq 1 \text{ for } i = 1, 2, \ldots, m; z^T y_j \leq 0 \text{ for } j = 1, 2, \ldots, k \},$$

and conversely. Particularly, if $C$ is a polyhedral cone, then

$$C^* = \{ z \in \mathbb{R}^n : z^T x \leq 0 \text{ for all } x \in C \}.$$