Exercises

Approximation Algorithms

Spring 2010

Sheet 5

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 1

Consider MULTI CONSTRAINT KNAPSACK where n objects with profits $p_i \in \mathbb{Q}_+$ and budget requirement $a_i^j \in [0,1]$ are given. Let

$$OPT = \max_{I \subseteq \{1,\dots,n\}} \Big\{ \sum_{i \in I} p_i \mid \sum_{i \in I} a_i^j \leq 1 \ \forall j = 1,\dots,k \Big\}$$

(since after scaling we may assume that the available budgets are all 1). We consider the number k of budgets to be a fixed constant. Show that for any $\varepsilon > 0$ one can compute in time polynomial in n and $1/\varepsilon$ a solution $I \subseteq \{1,\dots,n\}$ with profit $\sum_{i \in I} p_i \geq OPT$ which is *nearly feasible*, i.e. $\sum_{i \in I} a_i^j \leq 1 + \varepsilon$ for all $j = 1,\dots,k$.

Hints: Find a suitable rounding + dynamic programming.

Solution:

Suppose $\frac{1}{\varepsilon} \in \mathbb{Z}$. Round all a_i^j *down* to the nearest multiple of $\frac{\varepsilon}{n}$ and denote the new numbers with $\overline{a}_i^j \in \mathbb{Z}\frac{\varepsilon}{n}$. Let OPT' be the best value of the new MCK instance. Due to the rounding down $OPT' \geq OPT$.

For any $b_1,\dots,b_k \in \{0, \frac{\varepsilon}{n}, 2\frac{\varepsilon}{n}, \dots, 1\}$ consider table entries

$$\begin{aligned} A(m, b_1, \dots, b_k) &= \text{best profit from a subset of items } \{1,\dots,m\} \text{ such that} \\ &\quad\ \text{budget of type } j \text{ is} \leq b_j \ \forall j = 1,\dots,k \\ &= \max_{I \subseteq \{1,\dots,m\}} \Big\{ \sum_{i \in I} p_i \mid \sum_{i \in I} a_i^j \leq b_j \ \forall j = 1,\dots,k \Big\} \end{aligned}$$

Clearly the number of table entries is $O(n^{k+1}/\varepsilon^k)$. We compute a single entry by

$$A(m, b_1, \dots, b_k) = \max\Big\{ A(m-1, b_1, \dots, b_k), A(m-1, b_1 - \overline{a}_m^1, \dots, b_k - \overline{a}_m^k) + p_m \Big\}$$

We can compute each entry in $O(1)$. Finally the solution I leading to

$$OPT' = A(n, 1, \dots, 1)$$

can be easily reconstructed. It has profit $\sum_{i \in I} p_i = OPT' \geq OPT$. It might violate some budgets, but

$$\sum_{i \in I} a_i^j \leq \sum_{i \in I} (\overline{a}_i^j + \frac{\varepsilon}{n}) \leq \underbrace{\sum_{i \in I} \overline{a}_i^j}_{\leq 1} + n \cdot \frac{\varepsilon}{n} \leq 1 + \varepsilon$$

Exercise 2

For the BIN COVERING problem, we are given an instance $I = (a_1, \ldots, a_n)$ with items $a_i \in [0,1]$ and aim at maximizing the number of bins, which are *covered* (that means the size of the assigned items is at least 1):

$$OPT = \max\left\{k \mid \exists I_1 \dot\cup \ldots \dot\cup I_k = \{1,\ldots,n\} : \forall j : \sum_{i\in I_j} a_i \geq 1\right\}$$

We assume that $a_i \geq \delta, i = 1,\ldots,n$ for a constant $\delta > 0$. Give an asymptotic PTAS for this problem under the above assumption (i.e. a polynomial time algorithm that for any fixed $\varepsilon > 0$, covers at least $(1-\varepsilon)OPT - O(1)$ bins).

Hint: Adapt the APTAS of Fernandez de la Vega & Lueker from the lecture. In the grouping, you should round *down* the item sizes (instead of rounding them up).

Solution:

We apply the following grouping strategy

- INPUT: Instance $I = (a_1, \ldots, a_n)$, $k \in \mathbb{N}$
- OUTPUT: Instance $I' = (a'_1, \ldots, a'_n)$ with $a'_i \leq a_i$ and $\leq k$ different item sizes

Suppose for simplicity $n/k \in \mathbb{Z}$.

(1) Sort $a_1 \leq a_2 \leq \ldots \leq a_n$

(2) Partition items into k consecutive groups of $\lceil n/k \rceil$ items (the last group might have less items)

(3) Let a'_i be the size of the smallest item in i's group

First of all, any solution for I' gives a solution to I with the same number of bins. We claim that the other way around

Claim: $OPT(I') \geq OPT(I) - \lceil n/k \rceil$.

Proof: To see this consider any solution to I. For every item in group j, we find an item of I' in group $j+1$ (which is at least as big). This way we again cover all bins that did not contain any item from the last group. But these are at most n/k many.

Claim: For an instance I' with $a_i \geq \delta$ and $1/\varepsilon^2$ many different item sizes, one can find an optimum BIN COVERING solution in polynomial time (considering ε, δ as constant).

Proof: In the optimum solution w.l.o.g. no pattern contains more than $2/\delta$ items, since then one could split it into 2 bins, each containing $1/\delta$ items of size $\geq \delta$ each. Hence there are $O((1/\varepsilon^2)^{1/\delta})$ many different patterns and $n^{O((1/\varepsilon^2)^{1/\delta})}$ possible solutions that one has to check.

We consider the following algorithm

1. Choose $k := 1/\varepsilon^2$
2. Apply linear grouping to $I \to I'$
3. Find an optimum solution to I'

Assume $\varepsilon \leq \delta$. Cleary $OPT \geq \frac{\delta n}{2} \geq \frac{\varepsilon n}{2}$ (again since $a_i \geq \delta$ and in OPT never more than $2/\delta$ items are assigned to the same bin). Using the grouping we produce a solution of value

$$OPT(I') \geq OPT - \lceil \varepsilon^2 n \rceil = OPT - \lceil 2\varepsilon \cdot \underbrace{\frac{\varepsilon n}{2}}_{\leq OPT} \rceil \geq (1-2\varepsilon)OPT - 1.$$