Integer Points in Polyhedra

Spring 2009

Assignment Sheet 5

Exercise 1 (Bin-packing problem)
Consider the following linear program:

$$\begin{aligned} \min \ & \sum_{i=1}^{t} \lambda_i \\ \text{s.t.} \ & \sum_{i=1}^{t} \lambda_i v_i = b, \\ & \lambda_i \geq 0, \end{aligned}$$

where $b \in \mathbb{Z}^d$ is a given integral vector, $v_1, v_2, \ldots, v_t$ are all integral solutions of a given knapsack problem

$$a^{\mathrm{T}} v \leq \beta, \qquad v \geq 0.$$

Show that this linear program can be solved in polynomial time if d is fixed.

Exercise 2 (Lattice points in a knapsack)
Derive an upper bound on the number of vertices of $\operatorname{conv}(K \cap \Lambda)$, where Λ is a lattice and

$$K = \left\{x : a^{\mathrm{T}} x \leq \beta,\ x \geq 0\right\}$$

for some vector a and some number β.

Exercise 3 (Computation of the integer hull in fixed dimension)
Describe an efficient algorithm to compute the integer hull of a given rational polyhedron in fixed dimension.

Exercise 4 (Diophantine approximation)
We consider the following problem: Given n numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\varepsilon > 0$, find a “small” positive integer q and integers $p_1, p_2, \ldots, p_n$ such that

$$|\alpha_i q - p_i| < \varepsilon, \qquad i = 1, 2, \ldots, n,$$

or equivalently,

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\varepsilon}{q}, \qquad i = 1, 2, \ldots, n,$$

(a) Show that there are integers $p_1, p_2, \ldots, p_n$ and q such that

$$0 < q \le \varepsilon^{-n}$$

and

$$|\alpha_i q - p_i| < \varepsilon, \qquad i = 1, 2, \ldots, n.$$

(b) Show that there is a polynomial algorithm that computes integers $p_1, p_2, \ldots, p_n$ and q such that

$$0 < q \le 2^{\frac{n(n+1)}{4}} \varepsilon^{-n}$$

and

$$|\alpha_i q - p_i| < \varepsilon, \qquad i = 1, 2, \ldots, n.$$