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Exercises

Approximation Algorithms

Spring 2010

Sheet 3

Exercise 1
Consider the k-SET COVERING problem: Given a family of sets $S_1,\dots,S_m \subseteq U$ of cardinality $|S_i| \leq k$ with cost $c(S_i)$, find a subset of these sets that minimize the cost, while each element has to be covered at least once. Recall the linear programming relaxation

$$\begin{array}{rcll} \min \sum_{i=1}^{m} c(S_i)\cdot x_i & & & (ILP) \\ \sum_{i: j\in S_i} x_i & \geq & 1 & \forall j \in U \\ x_i & \geq & 0 & \forall i \end{array}$$

where x_i indicates, whether to take set S_i.

i) Let x^* be an optimum **basic** solution for (LP). Prove that there is an i with $x_i^* \geq \frac{1}{k}$.

ii) Consider the following *iterative rounding algorithm*:

(1) WHILE $U \neq \emptyset$ DO
  (2) Compute an optimum basic solution x^*
  (3) Choose i with $x_i^* \geq \frac{1}{k}$
  (4) Buy set S_i, delete elements in S_i from the instance
(5) Output bought sets

Prove that this algorithm gives a k-approximation.
Hint: How much does the value of the optimum fractional solution decrease in each iteration compared to the bought set?

Exercise 2
For the STEINER TREE problem, we are given an undirected weighted graph $G = (V,E)$ with a cost function $c : E \to \mathbb{Q}_+$ and a set of terminals $R \subseteq V$. It is the goal to find a tree T that connects all terminals. A natural linear programming relaxation is

$$\begin{array}{rcll} \min \sum_{e\in E} c_e x_e & & & (LP) \\ \sum_{e\in\delta(S)} x_e & \geq & 1 & \forall S \subseteq V : 1 \leq |S\cap R| < |R| \\ x_e & \geq & 0 & \forall e \in E \end{array}$$

Here $\delta(S) = \{\{u,v\} \in E \mid u \in S, v \notin S\}$ are the edges, crossing S. Show that one can compute an optimum fractional solution for (LP) in polynomial time (to be precise: Show that the LP can be solved in time polynomial in $n = |V|$ and the encoding length $\langle c \rangle$ of c).
Hint: Use the Ellipsoid method from the lecture. Recall that the s-t MINCUT problem is polynomial time solvable: Given a graph $G = (V,E)$, nodes $s,t \in V$ and capacities $w : E \to \mathbb{Q}_+$, compute an s-t cut $S \subseteq V$ with $s \in S, t \notin S$ that minimizes $\sum_{e \in \delta(S)} w(e)$.